J Stat Phys (2008) 133: 841–870
DOI 10.1007/s10955-008-9636-6

Equilibrium Solution to the Inelastic Boltzmann Equation Driven by a Particle Bath

Marzia Bisi · José A. Carrillo · Bertrand Lods

Received: 21 May 2008 / Accepted: 10 October 2008 / Published online: 30 October 2008

Abstract We show the existence of smooth stationary solutions for the inelastic Boltzmann equation under the thermalization induced by a host medium with a fixed distribution. This is achieved by controlling the L^p-norms, the moments and the regularity of the solutions to the Cauchy problem together with arguments related to a dynamical proof for the existence of stationary states.

1 Introduction

The dynamics of rapid granular flows is commonly modelled by a suitable modification of the Boltzmann equation for inelastic hard-spheres interacting through binary collisions [21, 45]. As well-known, in absence of energy supply, inelastic hard spheres are cooling down and the energy continuously decreases in time. In particular, the Boltzmann collision operator for inelastic hard spheres admits only trivial equilibria. This is no more the case if the spheres are forced to interact with an external agency (thermostat) and, in such a case, the energy supply may lead to a non-equilibrium steady state. For such driven system (in a space homogeneous setting), the time-evolution of the one-particle distribution function $f(v,t)$, $v \in \mathbb{R}^3$, $t > 0$ satisfies the following

$$\partial_t f = \tau \mathcal{Q}(f, f) + \mathcal{G}(f), \tag{1.1}$$

M. Bisi (✉)
Dipartimento di Matematica, Università di Parma, Viale G.P. Usberti 53/A, 43100 Parma, Italy
e-mail: marzia.bisi@unipr.it

J.A. Carrillo
ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain
e-mail: carrillo@mat.uab.es

B. Lods
Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal (Clermont-Ferrand 2), 63177 Aubière Cedex, France
e-mail: bertrand.lods@math.univ-bpclermont.fr

where $\tau \geq 0$ is a given constant, $\mathcal{Q}(f, f)$ is the inelastic Boltzmann collision operator, expressing the effect of binary collisions of particles, while $\mathcal{G}(f)$ models the forcing term.

There exist in the literature several physical possible choices for the forcing term $\mathcal{G}$ in order to avoid the cooling of the granular gas: stochastic heating, particles heating or scaled variables to study the cooling of granular systems and even a nonlinear forcing term given by the quadratic elastic Boltzmann operator has been taken into account [29]. These options have been studied first in the case of inelastic Maxwell models [5, 6, 10, 11, 14, 15, 17, 19, 23, 24, 44]. The most natural one is the pure diffusion thermal bath for which

$$\mathcal{G}(f) = \mu \Delta f \tag{1.2}$$

where $\mu > 0$ is a constant, studied in [30, 39] for hard-spheres. Such a forcing term corresponds to the physical situation in which granular beads receive random kicking in their velocity, like air-levitated disks [12]. Another example is the thermal bath with linear friction

$$\mathcal{G}(f) = \mu \Delta f + \lambda \,\mathrm{div}(vf), \tag{1.3}$$

where λ and μ are positive constants. We also have to mention the fundamental example of anti-drift forcing term which is related to the existence of self-similar solution to the inelastic Boltzmann equation:

$$\mathcal{G}(f) = -\kappa \,\mathrm{div}(vf), \quad \kappa > 0. \tag{1.4}$$

This problem has been treated in [37, 38, 41] for hard-spheres. For all the forcing terms given by (1.2), (1.3), (1.4) it is possible to prove the existence of a non-trivial stationary state $F \geq 0$ such that

$$\tau \mathcal{Q}(F, F) + \mathcal{G}(F) = 0.$$

Moreover, such a stationary state can be chosen to be smooth, i.e. $F \in C^\infty(\mathbb{R}^3)$. Finally, even if the uniqueness (in suitable class of functions) of such a stationary state is an open problem, it can be shown for all these models that, in the weakly elastic regime in which the restitution coefficient is close to unity, the stationary state is unique. For an exhaustive survey of the "state of art" on the mathematical results for the evolution of granular media see [45].

We are concerned here with a similar question when the forcing term $\mathcal{G}$ is given by a *linear scattering operator*. This corresponds to a situation in which the system of inelastic hard spheres is immersed into a so called *thermal bath of particles*, i.e. $\mathcal{G}$ is given by a linear Boltzmann collision operator of the form:

$$\mathcal{G}(f) = \mathcal{B}[f, \mathbf{F}_1]$$

where $\mathbf{F}_1$ stands for the distribution function of the host fluid and $\mathcal{B}[\cdot, \cdot]$ is a given collision operator for (elastic or inelastic) hard-spheres. The precise definition of $\mathcal{G}$ is given in Sect. 2.1.

This kinetic model has been derived by means of a suitable asymptotic limit of binary mixtures, for example in [16] for the elastic case in the Maxwell molecules frame. For hard-spheres interactions, the model has already been tackled for instance in [8, 9] in order to derive closed macroscopic equations for granular powders in a host medium. Let us also mention the work [7] that investigates the case of a particle bath made of *elastic* hard-spheres at thermodynamical equilibrium (i.e. $\mathbf{F}_1$ is a suitable Maxwellian). The deviations of the steady state (which is there assumed to exist) from the Gaussian state are analyzed numerically. For inelastic Maxwellian molecules, the existence of a steady state for a particle bath has been obtained in [23]. To our knowledge, the existence of a stationary solution of (1.1) for particle bath heating and *inelastic hard-spheres* is an open problem and it is the main aim of this paper.

Our strategy, inspired by several works in the kinetic theory of granular gases [30, 37] or for coagulation-fragmentation problems [4, 28], is based on a dynamic proof of the existence of stationary states, see [23, Lemma 7.3] for a review. The exact "fixed point theorem" used here is reported in Sect. 2.2. The identification of a suitable Banach space and of a convex subset that remains invariant during the evolution, is achieved by controlling moments and L^p-norms of the solutions. In Sect. 3, we present regularity properties of the gain part of both collision operators $\mathcal{Q}$ and $\mathcal{G}$ in (1.1). Then, in Sect. 4 we get at first uniform bounds for the moments and the Lebesgue norms; in addition, we prove the strong continuity of the semi-group associated to (1.1), and the existence and uniqueness of a solution to the Cauchy problem. All this material allows to obtain, in Sect. 5, existence of non-trivial stationary states. Finally, Sect. 6 contains the study of regularity of stationary solutions. Many technical estimates involving the quadratic dissipative operator $\mathcal{Q}(f,f)$ are based on results presented in [20, 37, 38, 43] and in the references therein, but their extension to the linear inelastic operator $\mathcal{G}(f)$ is not trivial at all for the following reasons. First, since $\mathcal{G}$ is not quadratic, it induces a lack of symmetry particularly relevant in the study of propagation of L^p-norms. Second, since the microscopic collision mechanism is affected by the mass ratio of the two involved media (thermal bath and granular material), *Povzner-like estimates* for $\mathcal{G}$ are not straightforward consequences of previous results from [30]. Along this work we have put our emphasis on the new technical difficulties stemming from the linear inelastic operator, although we have included a sketch of the ideas, strategy and proofs in [30, 42, 43] for completeness. Let us finally mention that our analysis also applies to linear scattering model which corresponds to the case $\tau = 0$. For such a linear Boltzmann operator, we obtain the existence of an equilibrium solution, generalizing the results of [34, 36, 44] to non-necessarily Maxwellian host distribution.

2 Preliminaries

Let us introduce the notations we shall use in the sequel. Throughout the paper we shall use the notation $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$. We denote, for any $\eta\in\mathbb{R}$, the Banach space

$$L^1_\eta = \left\{ f:\mathbb{R}^3\to\mathbb{R} \text{ measurable};\ \|f\|_{L^1_\eta} := \int_{\mathbb{R}^3} |f(v)|\langle v\rangle^\eta \,\mathrm{d}v < +\infty \right\}.$$

More generally we define the weighted Lebesgue space $L^p_\eta(\mathbb{R}^3)$ ($p\in[1,+\infty)$, $\eta\in\mathbb{R}$) by the norm

$$\|f\|_{L^p_\eta(\mathbb{R}^3)} = \left[\int_{\mathbb{R}^3} |f(v)|^p \langle v\rangle^{p\eta}\,\mathrm{d}v\right]^{1/p}.$$

The weighted Sobolev space $W^{k,p}_\eta(\mathbb{R}^3)$ ($p\in[1,+\infty)$, $\eta\in\mathbb{R}$ and $k\in\mathbb{N}$) is defined by the norm

$$\|f\|_{W^{k,p}_\eta(\mathbb{R}^3)} = \left[\sum_{|s|\le k} \|\partial^s_v f\|^p_{L^p_\eta}\right]^{1/p}$$

where ∂_v^s denotes the partial derivative associated with the multi-index $s \in \mathbb{N}^N$. In the particular case $p = 2$ we denote $H^k_\eta = W^{k,2}_\eta$. Moreover this definition can be extended to H^s_η for any $s \geq 0$ by using the Fourier transform.

2.1 The Kinetic Model

We assume the granular particles to be perfectly smooth hard spheres of mass $m = 1$ performing inelastic collisions. Recall that, as usual, the inelasticity of the collision mechanism is characterized by a single parameter, namely the coefficient of normal restitution $0 < \epsilon < 1$. To define the collision operator we write

$$\mathcal{Q}(f, f) = \mathcal{Q}^+(f, f) - \mathcal{Q}^-(f, f), \tag{2.1}$$

where the "loss" term $\mathcal{Q}^-(f, f)$ is

$$\mathcal{Q}^-(f, f)(v) = f(f * |v|), \tag{2.2}$$

and the "gain" term $\mathcal{Q}^+(f, f)$ is given by

$$\mathcal{Q}^+(f, f)(v) = \frac{1}{2\pi\epsilon^2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |q \cdot \mathrm{n}| f('\tilde{v}) f('\tilde{w})\, \mathrm{d}\mathrm{n}\, \mathrm{d}w,$$

where $q = v - w$ is the relative velocity, $\mathrm{n} \in \mathbb{S}^2$ is the unit vector in the direction of impact, and $('\tilde{v}, '\tilde{w})$ are the pre-collisional velocities that result in v and w after collision. They read as

$$'\tilde{v} = v - \frac{\zeta}{\epsilon}(q \cdot \mathrm{n})\mathrm{n}, \qquad '\tilde{w} = w + \frac{\zeta}{\epsilon}(q \cdot \mathrm{n})\mathrm{n}, \tag{2.3}$$

with $\zeta = \frac{1+\epsilon}{2}$ (notice that we always have $\frac{1}{2} < \zeta < 1$). The weak formulation of the gain term reads as

$$\int_{\mathbb{R}^3} \mathcal{Q}^+(f, f)(v)\psi(v)\, \mathrm{d}v = \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v) f(w)|q \cdot \mathrm{n}| \int_{\mathbb{S}^2} \psi(\tilde{v}')\, \mathrm{d}\mathrm{n}\, \mathrm{d}w\, \mathrm{d}v, \tag{2.4}$$

where $\psi(\cdot)$ is a suitable test function and $\tilde{v}'$ is a post-collisional velocity. We refer the reader to [22, Appendix] for more details on the weak and strong forms of inelastic collision operators. The collision transformation that puts v and w into correspondence with the post-collisional velocities can be expressed as follows:

$$\tilde{v}' = v - \zeta(q \cdot \mathrm{n})\mathrm{n}, \qquad \tilde{w}' = w + \zeta(q \cdot \mathrm{n})\mathrm{n}. \tag{2.5}$$

By making use of the following identity [13, 26],

$$\int_{\mathbb{S}^2} (\hat{q} \cdot \mathrm{n})_+ \, \varphi(\mathrm{n}(q \cdot \mathrm{n}))\, \mathrm{d}\mathrm{n} = \frac{1}{4}\int_{\mathbb{S}^2} \varphi\left(\frac{q - |q|\sigma}{2}\right) \mathrm{d}\sigma \tag{2.6}$$

for any function φ, with $\hat{q} = q/|q|$, we can rewrite the operator both in strong form as

$$\mathcal{Q}^+(f, f)(v) = \frac{1}{4\pi\epsilon^2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v - w| f('v) f('w)\, \mathrm{d}\sigma\, \mathrm{d}w,$$

where the pre-collisional velocities read as

$$'v = v + \frac{\zeta}{2\epsilon}(|q|\sigma - q), \qquad 'w = w - \frac{\zeta}{2\epsilon}(|q|\sigma - q), \tag{2.7}$$

and in weak form (that will be the main tool in the rest of the paper) as

$$\int_{\mathbb{R}^3} \mathcal{Q}^+(f,f)(v)\psi(v)\,\mathrm{d}v = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)f(w)|q|\int_{\mathbb{S}^2}\psi(v')\,\mathrm{d}\sigma\,\mathrm{d}w\,\mathrm{d}v, \tag{2.8}$$

with

$$v' = v + \frac{\zeta}{2}(|q|\sigma - q), \qquad w' = w - \frac{\zeta}{2}(|q|\sigma - q). \tag{2.9}$$

Using the symmetry that allows us to exchange v with w in the integrals we obtain the following symmetrized weak form

$$\int_{\mathbb{R}^3} \mathcal{Q}(f,f)(v)\psi(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)f(w)|q|\mathcal{A}_\zeta[\psi](v,w)\,\mathrm{d}w\,\mathrm{d}v, \tag{2.10}$$

where

$$\mathcal{A}_\zeta[\psi](v,w) = \frac{1}{4\pi}\int_{\mathbb{S}^2}(\psi(v') + \psi(w') - \psi(v) - \psi(w))\,\mathrm{d}\sigma. \tag{2.11}$$

The inelastic Boltzmann operator $\mathcal{Q}(f,f)$ satisfies the basic conservation laws of mass and momentum, obtained by taking $\psi = 1, v$ in the weak formulation (2.10), since $\mathcal{A}_\zeta[1] = \mathcal{A}_\zeta[v] = 0$. On the other hand, in the modelling of dissipative kinetic equations, conservation of energy does not hold. In fact, we obtain $\mathcal{A}_\zeta[|v|^2] = -\frac{1-\epsilon^2}{4}|v-w|^2$ from which we deduce

$$\int_{\mathbb{R}^3} \mathcal{Q}(f,f)(v)|v|^2\,\mathrm{d}v = -\frac{1-\epsilon^2}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-w|^3 f(v)f(w)\,\mathrm{d}v\,\mathrm{d}w, \tag{2.12}$$

where we observe the dissipation of kinetic energy. In the absence of any other source of energy, the system cools down as $t \to \infty$ following Haff's law as proved in [37].

As already said in Introduction, the forcing term $\mathcal{G}$ arising in the kinetic equation (1.1) is chosen to be a linear scattering operator, corresponding to the so called *particle bath heating*,

$$\mathcal{G}(f)(v) := \mathcal{L}(f)(v) = \frac{1}{2\pi\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|q\cdot \mathrm{n}|\left[e^{-2}f(v_\star)\mathbf{F}_1(w_\star) - f(v)\mathbf{F}_1(w)\right]\mathrm{d}w\,\mathrm{d}\mathrm{n} \tag{2.13}$$

where λ is the mean free path, $q = v - w$, and $v_\star$, $w_\star$ are the pre-collisional velocities which result, respectively, in v and w after collision. The collision mechanism related to the linear scattering operator is characterized by

$$(v - w)\cdot \mathrm{n} = -e(v_\star - w_\star)\cdot \mathrm{n}, \tag{2.14}$$

where $0 < e < 1$ is the *constant restitution coefficient* (possibly different from ϵ). Here, we will consider a similar separation of the operator into gain and loss terms, $\mathcal{L}(f) = \mathcal{L}^+(f) - \mathcal{L}^-(f)$, with obvious definitions. Here the host fluid is made of hard-spheres of mass m_1 (possibly different from the traced particles mass $m = 1$) and the distribution function $\mathbf{F}_1$ of the host fluid fulfills the following:

Assumption 2.1 $\mathbf{F}_1$ *is a nonnegative normalized distribution function with bulk velocity* $\mathbf{u}_1 \in \mathbb{R}^3$ *and temperature* $\Theta_1 > 0$. *Moreover,* $\mathbf{F}_1$ *is smooth in the following sense,*

$$\mathbf{F}_1 \in H^s_\delta(\mathbb{R}^3), \quad \forall s, \delta \geq 0$$

and of finite entropy $\int_{\mathbb{R}^3} \mathbf{F}_1(v) \log \mathbf{F}_1(v)\, \mathrm{d}v < \infty$.

Remark 2.2 It is well-known [2, Lemma 4] that for such $\mathbf{F}_1$ there exists some $\chi > 0$ such that

$$\nu(v) := \frac{1}{2\pi\lambda} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - w) \cdot \mathrm{n}| \mathbf{F}_1(w)\, \mathrm{d}w\, \mathrm{d}\mathrm{n} \geq \chi \sqrt{1 + |v|^2} \quad \forall v \in \mathbb{R}^3. \tag{2.15}$$

The finiteness of entropy is not necessary to get this inequality, but it shall be essential later on. A particular choice of the distribution function $\mathbf{F}_1$, corresponding to a host fluid at thermodynamical equilibrium, is the following Maxwellian distribution

$$\mathbf{F}_1(v) = \mathcal{M}_1(v) = \left(\frac{m_1}{2\pi\Theta_1}\right)^{3/2} \exp\left\{-\frac{m_1(v - \mathbf{u}_1)^2}{2\Theta_1}\right\}, \quad v \in \mathbb{R}^3. \tag{2.16}$$

Notice however that our approach remains valid for more general distribution function.

For particles of mass $m = 1$ colliding inelastically with particles of mass m_1, the restitution coefficient being constant, the expressions of the pre-collisional velocities $(v_\star, w_\star)$ may be written as [21, 44]

$$v_\star = v - 2\alpha \frac{1 - \beta}{1 - 2\beta} (q \cdot \mathrm{n})\, \mathrm{n}, \qquad w_\star = w + 2(1 - \alpha) \frac{1 - \beta}{1 - 2\beta} (q \cdot \mathrm{n})\, \mathrm{n},$$

where α is the mass ratio and β denotes the inelasticity parameter

$$\alpha = \frac{m_1}{1 + m_1} \in (0, 1), \qquad \beta = \frac{1 - e}{2} \in [0, 1/2).$$

The post-collisional velocities are given by

$$v^\star = v - 2\alpha(1 - \beta)\, (q \cdot \mathrm{n})\, \mathrm{n}, \qquad w^\star = w + 2(1 - \alpha)(1 - \beta)\, (q \cdot \mathrm{n})\, \mathrm{n}. \tag{2.17}$$

As for the quadratic operator, by making use of the identity (2.6) we can rewrite the linear operator as

$$\mathcal{L}(f)(v) = \frac{1}{4\pi\lambda} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |q| \left[e^{-2} f(\tilde{v}_\star) \mathbf{F}_1(\tilde{w}_\star) - f(v) \mathbf{F}_1(w)\right] \mathrm{d}w\, \mathrm{d}\sigma \tag{2.18}$$

with

$$\tilde{v}_\star = v - \alpha \frac{1 - \beta}{1 - 2\beta} (q - |q|\sigma), \qquad \tilde{w}_\star = w + (1 - \alpha) \frac{1 - \beta}{1 - 2\beta} (q - |q|\sigma).$$

For such a description, the post-collisional velocities are

$$\tilde{v}^\star = v - \alpha(1 - \beta)\, (q - |q|\sigma), \qquad \tilde{w}^\star = w + (1 - \alpha)(1 - \beta)\, (q - |q|\sigma). \tag{2.19}$$

We consider Eq. (1.1) in the weak form: for any regular $\psi = \psi(v)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(v,t)\psi(v)\,\mathrm{d}v = \frac{\tau}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v,t)f(w,t)|q|\mathcal{A}_{\varsigma}[\psi](v,w)\,\mathrm{d}w\,\mathrm{d}v + \frac{1}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q|f(v,t)\mathbf{F}_1(w)\mathcal{J}_e[\psi](v,w)\,\mathrm{d}v\,\mathrm{d}w \quad (2.20)$$

where

$$\mathcal{J}_e[\psi](v,w) = \frac{1}{2\pi}\int_{\mathbb{S}^2} |\hat{q}\cdot \mathrm{n}|\left(\psi(v^\star) - \psi(v)\right)\mathrm{d}\mathrm{n} = \frac{1}{4\pi}\int_{\mathbb{S}^2}\left(\psi(\tilde{v}^\star) - \psi(v)\right)\mathrm{d}\sigma.$$

2.2 Proof of Stationary States: Basic Tools and Strategy

As stated in the Introduction, the final purpose of this paper is to prove the existence of a non-trivial regular stationary solution $F \geq 0$ to (1.1). Namely, we look for $F \in L^1$, $F \geq 0$ such that

$$\tau\mathcal{Q}(F,F) + \mathcal{L}(F) = 0. \quad (2.21)$$

Remark 2.3 Notice that such a problem is trivial in the elastic case $\epsilon = 1$ and whenever $\mathbf{F}_1$ is the Maxwellian distribution (2.16). Indeed, in such a case, the Maxwellian equilibrium distribution $\mathcal{M}^\sharp$ of $\mathcal{L}$ provided by [34, 36, 44] is a stationary solution to (1.1) since $\mathcal{Q}(\mathcal{M}^\sharp, \mathcal{M}^\sharp) = 0$ (elastic Boltzmann equation) and $\mathcal{L}(\mathcal{M}^\sharp) = 0$.

The main ingredients are to show the existence of fixed points for the flow map at any time, and thus a continuity in time argument of the semi-group that allows to identify this one-parameter family of fixed points as a stationary point of the flow. Contraction estimates in Fourier-based distances were used in the Maxwellian case [10, 23] to derive the fixed points of the flow map at any given time. Moreover, due to the strict contraction of the distances they were unique. These contraction estimates are not available in the hard-spheres case but they can be substituted by the Tykhonov Fixed Point Theorem as previously done in [4, 28, 30, 37] for the existence part. Uniqueness of stationary solutions is an open issue in our case. The exact result that will be used can be summarized as:

Lemma 2.4 (Dynamic proof of stationary states [4, 28, 30, 37]) *Let Y be a Banach space and $(S_t)_{t\geq 0}$ be a continuous semi-group on Y such that*

i) *there exists Z a nonempty convex and weakly (sequentially) compact subset of Y which is invariant under the action of S_t (that is $S_t z \in Z$ for any $z \in Z$ and $t \geq 0$);*
ii) *S_t is weakly (sequentially) continuous on Z for any $t > 0$.*

Then there exists $z_0 \in Z$ which is stationary under the action of S_t (that is $S_t z_0 = z_0$ for any $t \geq 0$).

The strategy is therefore to identify a Banach space Y and a convex subset $Z \subset Y$ in order to apply the above result. To do so, one shall prove that

- for any $f_0 \in Y$, there is a solution $f \in \mathcal{C}([0,\infty), Y)$ to (1.1) with $f(t=0) = f_0$;
- the solution f is unique in Y and if $f_0 \in Z$ then $f(t) \in Z$ for any $t \geq 0$;
- the set Z is (weakly sequentially) compactly embedded into Y;

- solutions to (1.1) have to be (weakly sequentially) stable, i.e., for any sequence $(f_n)_n \subset \mathcal{C}([0,\infty), Y)$ of solutions to (1.1) with $f_n(t) \in Z$ for any $t \geq 0$, then, there is a subsequence $(f_{n_k})_k$ which converges weakly to some $f \in \mathcal{C}([0,\infty), Y)$ such that f is a solution to (1.1).

If all the above points are satisfied by the evolution problem (1.1), then one can apply Lemma 2.4 to the semi-group $(\mathcal{S}_t)_{t\geq 0}$ which to any $f_0 \in Y$ associates the unique solution $f(t) = \mathcal{S}_t f_0$ to (1.1). Moreover, the regularity properties of the gain part of the operators [37] shall provide us the needed regularity to show the existence of smooth stationary states.

3 Regularity of Gain Operators

We recall the following result, taken from [37, Theorem 2.5, Proposition 2.6] and based on [20, 35], on the regularity properties of the gain part operator $\mathcal{Q}^+(g, f)$ that we state here only for hard-spheres interactions in space dimension $N = 3$.

Proposition 3.1 (Regularity of the gain term $\mathcal{Q}^+$) *For all $s, \eta > 0$, we have*

$$\left\| \mathcal{Q}^+(g, f) \right\|_{H^{s+1}_\eta} \leq C(s, \eta, \epsilon) \left[\|g\|_{H^s_{\eta+2}} \|f\|_{H^s_{\eta+2}} + \|g\|_{L^1_{\eta+2}} \|f\|_{L^1_{\eta+2}} \right]$$

where the constant $C(s, \eta, \epsilon) > 0$ only depends on the restitution coefficient $\epsilon \in (0, 1]$, s and η. Moreover, for any $p \in [1,\infty)$ and $\delta > 0$, there exist $\theta \in (0, 1)$ and a constant $C_\delta > 0$, only depending on p, ϵ and δ, such that

$$\int_{\mathbb{R}^3} \mathcal{Q}^+(f, f) f^{p-1} \, \mathrm{d}v \leq C_\delta \|f\|_{L^1}^{1+p\theta} \|f\|_{L^p}^{p(1-\theta)} + \delta \|f\|_{L^1_2} \|f\|^p_{L^p_{1/p}}.$$

On the other hand, the linear operator $\mathcal{L}(f)$ is quite similar to the quadratic Boltzmann operator associated to hard-spheres interactions and constant restitution coefficient e by fixing one of the distributions. In fact, it is possible to obtain the following similar result:

Proposition 3.2 (Regularity of the gain term $\mathcal{L}^+$) *For all $s, \eta > 0$, we have*

$$\left\| \mathcal{L}^+(f) \right\|_{H^{s+1}_\eta} \leq C(s, \eta, e) \left[\|\mathbf{F}_1\|_{H^s_{\eta+2}} \|f\|_{H^s_{\eta+2}} + \|\mathbf{F}_1\|_{L^1_{\eta+2}} \|f\|_{L^1_{\eta+2}} \right] \tag{3.1}$$

where the constant $C(s, \eta, e) > 0$ only depends on the restitution coefficient $e \in (0, 1]$, s and η. Moreover, for any $p \in (1,\infty)$ and $\delta > 0$, there exist $q < p$ and a constant $K_\delta > 0$, only depending on p, e and δ, such that

$$\int_{\mathbb{R}^3} \mathcal{L}^+(f) f^{p-1} \, \mathrm{d}v$$
$$\leq K_\delta \|\mathbf{F}_1\|_{L^q} \|f\|_{L^p}^{p-1} \|f\|_{L^1} + \delta \left(\|\mathbf{F}_1\|_{L^1_2} \|f\|^p_{L^p_{1/p}} + \|\mathbf{F}_1\|_{L^p_{1/p}} \|f\|_{L^1_2} \|f\|^{p-1}_{L^p_{1/p}} \right). \tag{3.2}$$

Proof The proof of these two estimates relies on the same steps as in Sects. 2.2, 2.3 and 2.4 of [37], see also [43]. We need just to have the same basic estimates as in their case. We start with the proof of (3.1). An expression of the Fourier transform of $\mathcal{L}^+$ can be obtained as:

$$\mathcal{F}\left[\mathcal{L}^+(f)\right](\xi) := \int_{\mathbb{R}^3} \exp(-i\xi \cdot v) \mathcal{L}^+(f)(v) \, \mathrm{d}v = \frac{1}{4\pi\lambda} \int_{\mathbb{S}^2} \widehat{G}(\xi_+, \xi_-) \, \mathrm{d}\sigma$$

with $G(v,w)=|v-w|f(v)\mathbf{F}_1(w)$, $\widehat{G}$ its Fourier transform with respect to (v,w) and

$$\xi_+=(1-\alpha(1-\beta))\xi+\alpha(1-\beta)|\xi|\sigma, \qquad \xi_-=\alpha(1-\beta)\xi-\alpha(1-\beta)|\xi|\sigma.$$

With this expression at hand, it is immediate to generalize to $\mathcal{L}^+$ the regularity result in [37, Theorem 2.5, Proposition 2.6] giving (3.1).

Now, let us prove the second result. We first notice that, as in [3], the gain operator $\mathcal{L}^+$ admits an integral representation. Actually, even if it is assumed in [3] that $\mathbf{F}_1$ is given by the Maxwellian distribution (2.16), a careful reading of the calculations of [3] yields

$$\mathcal{L}^+ f(v)=\int_{\mathbb{R}^3} f(w)k(v,w)\,\mathrm{d}w, \tag{3.3}$$

where

$$k(v,w)=\frac{1}{2e^2\gamma^2|v-w|}\int_{V_2\cdot(w-v)=0}\mathbf{F}_1\left(v+V_2+\frac{1-2\overline{\gamma}}{2\gamma}(w-v)\right)\mathrm{d}V_2$$

with $\gamma=\alpha\frac{1-\beta}{1-2\beta}$ and $\overline{\gamma}=(1-\alpha)\frac{1-\beta}{1-2\beta}$. Arguing as in [37], we define the operator $\mathcal{T}$ related to the Radon transform:

$$\mathcal{T}\,:\,g\in L^1(\mathbb{R}^3,\mathrm{d}v)\mapsto \mathcal{T}g(v)=\frac{1}{|v|}\int_{z\perp v} g(\mu v+z)\,\mathrm{d}z$$

where $\mu=1-\frac{1-2\overline{\gamma}}{2\gamma}$. For any $h\in\mathbb{R}^3$, let τ_h denote the translation operator $\tau_h f(v)=f(v-h)$, for any $v\in\mathbb{R}^3$. Then, for any $g\in L^1(\mathbb{R}^3,\mathrm{d}v)$, one sees that

$$\begin{aligned}(\tau_w\circ\mathcal{T})(g)(v)&=\frac{1}{|v-w|}\int_{z\perp(v-w)} g(\mu(v-w)+z)\,\mathrm{d}z\\&=\frac{1}{|v-w|}\int_{z\perp(v-w)} g\left(v-w+z+\frac{1-2\overline{\gamma}}{2\gamma}(w-v)\right)\mathrm{d}z,\quad \forall v,w\in\mathbb{R}^3.\end{aligned}$$

Choosing $g=\tau_{-w}\mathbf{F}_1$ leads to the following expression of the kernel $k(v,w)$:

$$k(v,w)=\frac{1}{2e^2\gamma^2}[\tau_w\circ\mathcal{T}\circ\tau_{-w}](\mathbf{F}_1)(v),\quad v,w\in\mathbb{R}^3.$$

This previous computation is at the heart of the arguments of [37, Theorem 2.2], from which one gets a version of Lions' Theorem [31–33] for a suitable regularized cut-off kernel with collision frequency of the form $B_{m,n}(|q|,\hat{q}\cdot\sigma)=\Phi_{S_n}(|q|)b_{S_m}(\hat{q}\cdot\sigma)$, with Φ_{S_n} smooth and with compact support $[\frac{2}{n},n]$, and b_{S_m} smooth and supported in $[-1+\frac{2}{m},1-\frac{2}{m}]$. More precisely, defining the smoothed-out operator in angular and radial variables $\mathcal{L}^+_{S_{m,n}}$ as in [37, Sect. 2.4]:

$$\mathcal{L}^+_{S_{m,n}}(f)=\frac{1}{4\pi\lambda e^2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_{m,n}(|q|,\hat{q}\cdot\sigma)f(v_\star)\mathbf{F}_1(w_\star)\,\mathrm{d}w\,\mathrm{d}\sigma \tag{3.4}$$

then, for any $\eta\in\mathbb{R}^+$ and any $p>1$, there is $C(p,\eta,m,n)>0$ depending only on p, η and (m,n), such that

$$\|\mathcal{L}^+_{S_{m,n}}(f)\|_{L^p_\eta}\leq C(p,\eta,m,n)\|\mathbf{F}_1\|_{L^q_\eta}\|f\|_{L^1_{2|\eta|}} \tag{3.5}$$

for some $q < p$ given by $q = \frac{5p}{3+2p}$ if $p \in (1, 6]$ while $q = \frac{p}{3}$ if $p \in [6, +\infty)$ (see [37, Corollary 2.4]). In particular, Hölder's inequality leads to

$$\int_{\mathbb{R}^3} \mathcal{L}^+_{S_{m,n}}(f) f^{p-1} \,\mathrm{d}v \le \left(\int_{\mathbb{R}^3} f^p \,\mathrm{d}v \right)^{\frac{p-1}{p}} \|\mathcal{L}^+_{S_{m,n}}(f)\|_{L^p} \le C(m,n) \|f\|_{L^1} \|\mathbf{F}_1\|_{L^q} \|f\|^{p-1}_{L^p}$$

for some explicit constant $C(m,n) > 0$.

Similarly, one can define the remainder part of $\mathcal{L}^+$ which splits as

$$\mathcal{L}^+ - \mathcal{L}^+_{S_{m,n}} =: \mathcal{L}^+_{R_{m,n}} = \mathcal{L}^+_{RS_{m,n}} + \mathcal{L}^+_{SR_{m,n}} + \mathcal{L}^+_{RR_{m,n}}$$

with

$$\mathcal{L}^+_{RS_{m,n}}(f) = \frac{1}{4\pi\lambda e^2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_{R_n}(|q|) b_{S_m}(\hat{q} \cdot \sigma) f(v_\star) \mathbf{F}_1(w_\star) \,\mathrm{d}w \,\mathrm{d}\sigma,$$
$$\mathcal{L}^+_{SR_{m,n}}(f) = \frac{1}{4\pi\lambda e^2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_{S_n}(|q|) b_{R_m}(\hat{q} \cdot \sigma) f(v_\star) \mathbf{F}_1(w_\star) \,\mathrm{d}w \,\mathrm{d}\sigma,$$
$$\mathcal{L}^+_{RR_{m,n}}(f) = \frac{1}{4\pi\lambda e^2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_{R_n}(|q|) b_{R_m}(\hat{q} \cdot \sigma) f(v_\star) \mathbf{F}_1(w_\star) \,\mathrm{d}w \,\mathrm{d}\sigma,$$

where $\Phi_{R_n}(|q|) = |q| - \Phi_{S_n}(|q|)$ and $b_{R_m}(\hat{q} \cdot \sigma) = 1 - b_{S_m}(\hat{q} \cdot \sigma)$, $q \in \mathbb{R}^3$, $\sigma \in \mathbb{S}^2$. Hölder's inequality provides

$$\int_{\mathbb{R}^3} \mathcal{L}^+_{R_{m,n}}(f) f^{p-1} \,\mathrm{d}v \le \|f\|^{p-1}_{L^p_{1/p}} \|\mathcal{L}^+_{R_{m,n}}(f)\|_{L^p_{-1/p'}}$$

with p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, hence we have to estimate L^p_η norms of $\mathcal{L}^+_{SR_{m,n}}$, $\mathcal{L}^+_{RS_{m,n}}$, $\mathcal{L}^+_{RR_{m,n}}$ for $\eta = -1/p'$.

One can easily use [37, Theorem 2.1] to prove that, for any $\eta \in \mathbb{R}$,

$$\|\mathcal{L}^+_{SR_{m,n}}(f) + \mathcal{L}^+_{RR_{m,n}}(f)\|_{L^p_\eta} \le \varepsilon(m) \left(\|\mathbf{F}_1\|_{L^1_{|1+\eta|+|\eta|}} \|f\|_{L^p_{1+\eta}} + \|f\|_{L^1_{|1+\eta|+|\eta|}} \|\mathbf{F}_1\|_{L^p_{1+\eta}} \right)$$

for some explicit constant $\varepsilon(m)$ that, since the angular part of the collision kernel is such that $\lim_{m\to\infty} \|b_{R,m}\|_{L^1(\mathbb{S}^2)} = 0$, converges to 0 as m goes to infinity.

It remains to estimate the norm of $\mathcal{L}^+_{RS_{m,n}}(f)$. We follow now the lines of [42, Chap. 9, p. 395] (which differs slightly from [37, Proposition 2.6] and is more adapted to the linear case). Precisely, we split f as $f = f_r + f_{r^c} = f(v)\chi_{\{|v| \le r\}} + f(v)\chi_{\{|v|>r\}}$ for some $r > 0$. Then, as in [42, p. 395], there is some positive constant $C > 0$ such that

$$\|\mathcal{L}^+_{RS_{m,n}}(f_r)\|_{L^p_\eta} \le C \frac{r}{n} \|\mathbf{F}_1\|_{L^1_{|2+\eta|+|\eta|}} \|f\|_{L^p_{1+\eta}}$$

while

$$\|\mathcal{L}^+_{RS_{m,n}}(f_{r^c})\|_{L^p_\eta} \le C \frac{m^\lambda}{r} \|f\|_{L^1_{|2+\eta|+|\eta|}} \|\mathbf{F}_1\|_{L^p_{1+\eta}}$$

with $\lambda > 0$.

Gathering all the above estimates we get, for $\eta = -1/p'$,

$$\int_{\mathbb{R}^3} \mathcal{L}^+_{R_{m,n}}(f) f^{p-1}\,\mathrm{d}v \le C\|f\|^{p-1}_{L^p_{1/p}} \left(\frac{r}{n}\|\mathbf{F}_1\|_{L^1_2}\|f\|_{L^p_{1/p}} + \frac{m^\lambda}{r}\|f\|_{L^1_2}\|\mathbf{F}_1\|_{L^p_{1/p}} \right)$$
$$+ \varepsilon(m)\left(\|\mathbf{F}_1\|_{L^1_1}\|f\|^p_{L^p_{1/p}} + \|\mathbf{F}_1\|_{L^p_{1/p}}\|f\|_{L^1_1}\|f\|^{p-1}_{L^p_{1/p}} \right)$$
$$\le \left(C\frac{r}{n} + \varepsilon(m) \right)\|\mathbf{F}_1\|_{L^1_2}\|f\|^p_{L^p_{1/p}}$$
$$+ \left(C\frac{m^\lambda}{r} + \varepsilon(m) \right)\|\mathbf{F}_1\|_{L^p_{1/p}}\|f\|_{L^1_2}\|f\|^{p-1}_{L^p_{1/p}}.$$

The proof follows then by choosing first m large enough then r large enough and subsequently n big enough. □

4 Regularity Estimates for the Cauchy Problem

4.1 Nonnegativeness of the Solution

Let us consider the Cauchy problem

$$\partial_t f = \tau \mathcal{Q}(f, f) + \mathcal{L}(f) \quad (t > 0), \qquad f(t = 0) = f_0 \tag{4.1}$$

where $f_0 = f_0(v)$ is a given initial state, $f_0 \in L^1(\mathbb{R}^3)$. For any $f \in L^1$, let

$$\Sigma(f)(v) = \tau(|\cdot| * f)(v) + \nu(v) = \tau \int_{\mathbb{R}^3} |v - w| f(w)\,\mathrm{d}w + \nu(v).$$

With the notations of the previous section, it is easy to see that any solution $f(t)$ to (4.1) is given by the following Duhamel representation:

$$f(v,t) = f_0(v)\mathrm{e}^{-\int_0^t \Sigma(f)(v,s)\,\mathrm{d}s} + \int_0^t \left(\tau \mathcal{Q}^+(f, f) + \mathcal{L}^+(f)\right)(v,s)\mathrm{e}^{-\int_s^t \Sigma(f)(v,r)\,\mathrm{d}r}\,\mathrm{d}s. \tag{4.2}$$

Since $\mathcal{Q}^+(g, g) \ge 0$ and $\mathcal{L}^+(g) \ge 0$ for any nonnegative $g \in L^1(\mathbb{R}^3)$, it is easy to deduce in a very standard way, see for instance [1], that the solution $f(v,t)$ given by (4.2) is *nonnegative* for any $t \ge 0$ provided f_0 is, namely

$$f_0(v) \ge 0 \implies f(v,t) \ge 0 \quad \forall t > 0,\ v \in \mathbb{R}^3.$$

4.2 Evolution of Mean Velocity and Temperature

Let $f(v,t)$ be a nonnegative solution to (1.1). Define the mass density, the bulk velocity

$$\varrho(t) = \int_{\mathbb{R}^3} f(v,t)\,\mathrm{d}v, \qquad \mathbf{u}(t) = \frac{1}{\varrho(t)} \int_{\mathbb{R}^3} v f(v,t)\,\mathrm{d}v$$

and the temperature

$$\Theta(t) = \frac{1}{3\varrho(t)} \int_{\mathbb{R}^3} |v - \mathbf{u}(t)|^2 f(v,t)\,\mathrm{d}v, \quad \forall t \ge 0.$$

Note that (2.20) for $\psi = 1$ leads to the mass conservation identity $\dot{\varrho}(t) = 0$ i.e.

$$\varrho(t) = \varrho(0) := 1.$$

Now, (2.20) for $\psi(v) = v$ yields

$$\dot{\mathbf{u}}(t) = -\frac{\alpha(1-\beta)}{\lambda} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-w|(v-w) f(v,t) \mathbf{F}_1(w)\, \mathrm{d}v\, \mathrm{d}w, \quad \forall t \geq 0$$

which illustrates the fact that the bulk velocity is not conserved. To estimate the second order moment of f, let us introduce the auxiliary function:

$$F(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-w|^2 f(v,t) \mathbf{F}_1(w)\, \mathrm{d}v\, \mathrm{d}w.$$

Notice that

$$F(t) = \int_{\mathbb{R}^3} |v-\mathbf{u}_1|^2 f(v,t)\, \mathrm{d}v + \frac{3}{m_1}\Theta_1 = 3\Theta(t) + |\mathbf{u}(t)-\mathbf{u}_1|^2 + \frac{3}{m_1}\Theta_1. \tag{4.3}$$

In particular, to obtain uniform in time bounds of the mean velocity and the temperature, it is enough to provide uniform in time estimates of $F(t)$. With the special choice $\psi(v) = |v-\mathbf{u}_1|^2$ one has

$$\mathcal{A}_\zeta[\psi](v,w) = \frac{\zeta(1-\zeta)|q|}{4\pi} \int_{\mathbb{S}^2} (\sigma \cdot q - |q|)\, \mathrm{d}\sigma = -\zeta(1-\zeta)|q|^2 = -\frac{1-\epsilon^2}{4}|q|^2$$

while

$$\begin{aligned} \mathcal{J}_e[\psi](v,w) &= 2\alpha^2(1-\beta)^2|q|^2 - 2\alpha(1-\beta)\langle q, v-\mathbf{u}_1\rangle \\ &= -2\kappa(1-\kappa)|q|^2 - 2\kappa\langle q, w-\mathbf{u}_1\rangle, \quad v,w \in \mathbb{R}^3 \end{aligned}$$

with $\kappa = \alpha(1-\beta) = \frac{\alpha}{2}(1+e) \in (0,1)$ and $\langle \cdot,\cdot \rangle$ denoting the scalar product. It is easy to see that

$$\begin{aligned} \dot{F}(t) = &-\frac{(1-\epsilon^2)\tau}{8} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v,t) f(w,t)|q|^3\, \mathrm{d}v\, \mathrm{d}w \\ &- \frac{2\kappa(1-\kappa)}{\lambda} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-w|^3 f(v,t)\mathbf{F}_1(w)\, \mathrm{d}v\, \mathrm{d}w \\ &+ \frac{2\kappa}{\lambda} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q|\langle q, \mathbf{u}_1 - w\rangle f(v,t)\mathbf{F}_1(w)\, \mathrm{d}v\, \mathrm{d}w. \end{aligned} \tag{4.4}$$

Now, since $\int_{\mathbb{R}^3} f(v,t)\, \mathrm{d}v = 1$ for any $t \geq 0$, Jensen's inequality yields

$$\int_{\mathbb{R}^3} f(w,t)|q|^3\, \mathrm{d}w \geq \left| v - \int_{\mathbb{R}^3} w f(w,t)\, \mathrm{d}w \right|^3 = |v-\mathbf{u}(t)|^3$$

and consequently

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v,t)f(w,t)|q|^3\,\mathrm{d}v\,\mathrm{d}w \ge \int_{\mathbb{R}^3}|v-\mathbf{u}(t)|^3 f(v,t)\,\mathrm{d}v$$

$$\ge \left(\int_{\mathbb{R}^3}|v-\mathbf{u}(t)|^2 f(v,t)\,\mathrm{d}v\right)^{3/2} = (3\Theta(t))^{3/2}$$

where we used again Jensen's inequality. In the same way,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|q|^3 f(v,t)\mathbf{F}_1(w)\,\mathrm{d}v\,\mathrm{d}w \ge \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-w|^2 f(v,t)\mathbf{F}_1(w)\,\mathrm{d}v\,\mathrm{d}w\right)^{3/2} = F(t)^{3/2}.$$

Finally, the third integral in (4.4) is estimated as

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|q|\langle q,\mathbf{u}_1-w\rangle f(v,t)\mathbf{F}_1(w)\,\mathrm{d}v\,\mathrm{d}w\\
&\quad\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|q|^2|\mathbf{u}_1-w|f(v,t)\mathbf{F}_1(w)\,\mathrm{d}v\,\mathrm{d}w\\
&\quad\le 2\int_{\mathbb{R}^3}|v-\mathbf{u}_1|^2 f(v,t)\,\mathrm{d}v\int_{\mathbb{R}^3}|w-\mathbf{u}_1|\mathbf{F}_1(w)\,\mathrm{d}w + 2\int_{\mathbb{R}^3}|w-\mathbf{u}_1|^3\mathbf{F}_1(w)\,\mathrm{d}w\\
&\quad\le C_0F(t)
\end{aligned}$$

where

$$C_0 = 2\max\left\{\int_{\mathbb{R}^3}|w-\mathbf{u}_1|\mathbf{F}_1(w)\,\mathrm{d}w,\ \frac{\int_{\mathbb{R}^3}|w-\mathbf{u}_1|^3\mathbf{F}_1(w)\,\mathrm{d}w}{\int_{\mathbb{R}^3}|w-\mathbf{u}_1|^2\mathbf{F}_1(w)\,\mathrm{d}w}\right\}.$$

In conclusion, we obtain

$$\dot{F}(t) \le -\frac{(1-\epsilon^2)\tau}{8}(3\Theta(t))^{3/2} - \frac{2\kappa(1-\kappa)}{\lambda}F(t)^{3/2} + \frac{2C_0\kappa}{\lambda}F(t) \le -\gamma_1 F(t)^{3/2} + \gamma_2 F(t) \tag{4.5}$$

where $\gamma_1 = \frac{2\kappa(1-\kappa)}{\lambda} > 0$ and $\gamma_2 = \frac{2C_0\kappa}{\lambda} > 0$. A simple use of the maximum principle shows that

$$F(t) \le \max\left\{\left(\frac{\gamma_2}{\gamma_1}\right)^2, F(0)\right\}, \quad \forall t \ge 0.$$

Because of (4.3), this leads to explicit upper bounds of the temperature $\Theta(t)$ and the velocity $|\mathbf{u}(t)-\mathbf{u}_1|$, namely

$$\sup_{t\ge 0}\left(3\Theta(t) + |\mathbf{u}(t)-\mathbf{u}_1|^2\right) \le \max\left\{\left(\frac{\gamma_2}{\gamma_1}\right)^2, F(0)\right\} < \infty. \tag{4.6}$$

4.3 Propagation of Moments

To extend the previous basic estimates, in the spirit of [18], we deduce from Povzner-like estimates some useful inequalities on the moments

$$\mathbb{Y}_r(t) = \int_{\mathbb{R}^3} f(v,t)|v|^{2r}\,\mathrm{d}v, \quad t \ge 0,\ r \ge 1$$

where $f(t)$ is a solution to (1.1) with unit mass. One sees from (1.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{Y}_r(t) = \tau Q_r(t) + L_r(t),$$

where

$$Q_r(t) = \int_{\mathbb{R}^3} \mathcal{Q}(f,f)(v,t)|v|^{2r}\,\mathrm{d}v, \qquad L_r(t) = \int_{\mathbb{R}^3} \mathcal{L}(f)(v,t)|v|^{2r}\,\mathrm{d}v.$$

The calculations provided in [18, 30] allow to estimate, in an almost optimal way, the quantity Q_r. One has to do the same for $L_r(t)$ given by

$$L_r(t) = \frac{1}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v,t)\mathbf{F}_1(w)|v-w|\mathcal{J}_e[|\cdot|^{2r}](v,w)\,\mathrm{d}v\,\mathrm{d}w.$$

To do so, let us derive *Povzner-like estimates* for $\mathcal{L}$ in the spirit of [30]. The application of the result of [30] is not straightforward since, obviously, $\mathcal{L}$ is not quadratic and because of the influence of the mass ratio $\alpha = \frac{m_1}{m+m_1}$ in the collision mechanism. Here, we will write the mass of particles m even if taken as unity for the sake of the reader. To be precise, we are looking for estimates of

$$\mathcal{J}_e[|\cdot|^{2r}](v,w) = \frac{1}{2\pi}\int_{\mathbb{S}^2} |\hat{q}\cdot \mathrm{n}|\left(|v^\star|^{2r} - |v|^{2r}\right)\mathrm{dn}, \quad r \geq 1.$$

To do so, it shall be convenient to write

$$\mathcal{J}_e[|\cdot|^{2r}](v,w) = \frac{1}{2\pi m^r}\int_{\mathbb{S}^2} |\hat{q}\cdot \mathrm{n}|\left\{\Psi\left(m|v^\star|^2\right) - \Psi\left(m|v|^2\right)\right\}\mathrm{dn} \tag{4.7}$$

where $\Psi(x) = x^r$, $r \geq 1$. We adopt the strategy used in [30] and write

$$\Psi\left(m|v^\star|^2\right) - \Psi\left(m|v|^2\right) = \mathfrak{q}_e(\Psi)(v,w) + \Psi(m_1|w|^2) - \Psi(m_1|w^\star|^2) \tag{4.8}$$

where

$$\mathfrak{q}_e(\Psi)(v,w) = \Psi\left(m|v^\star|^2\right) + \Psi\left(m_1|w^\star|^2\right) - \Psi\left(m|v|^2\right) - \Psi\left(m_1|w|^2\right).$$

Now,

$$\mathfrak{q}_e(\Psi)(v,w) = \mathfrak{p}_e(\Psi)(v,w) - \mathfrak{n}_e(\Psi)(v,w)$$

with

$$\begin{cases}\mathfrak{p}_e(\Psi)(v,w) = \Psi\left(m|v|^2 + m_1|w|^2\right) - \Psi\left(m|v|^2\right) - \Psi\left(m_1|w|^2\right)\\ \mathfrak{n}_e(\Psi)(v,w) = \Psi\left(m|v|^2 + m_1|w|^2\right) - \Psi\left(m|v^\star|^2\right) - \Psi\left(m_1|w^\star|^2\right).\end{cases}$$

Applying [30, Lemma 3.1] to the function Ψ with $x = m|v|^2$ and $y = m_1|w|^2$, we see that there exists $A > 0$ such that

$$\mathfrak{p}_e(\Psi)(v,w) \leq A\Big(m|v|^2\Psi'\left(m_1|w|^2\right) + m_1|w|^2\Psi'\left(m|v|^2\right)\Big) \tag{4.9}$$

while, since Ψ is nondecreasing and $m|v|^2 + m_1|w|^2 \geq m|v^\star|^2 + m_1|w^\star|^2$, there exists $b > 0$ such that

$$\mathfrak{n}_e(\Psi)(v,w) \geq bmm_1|v^\star|^2|w^\star|^2\Psi''\left(m|v^\star|^2 + m_1|w^\star|^2\right).$$

One can then write

$$\mathfrak{n}_e(\Psi)(v,w) \geq b\Delta(v^\star,w^\star)\left(m|v^\star|^2+m_1|w^\star|^2\right)^2 \Psi''\left(m|v^\star|^2+m_1|w^\star|^2\right)$$

where

$$\Delta(v^\star,w^\star) = \frac{m|v^\star|^2 m_1|w^\star|^2}{(m|v^\star|^2+m_1|w^\star|^2)^2}.$$

To estimate better the above term $\Delta(v^\star,w^\star)$, it will be convenient to parametrize the post-collisional velocities in the *center of mass–relative velocity* variables, which, with respect to the usual transformation (see e.g. [30, (3.10)]) depend on the masses m and m_1. Namely, let us set

$$v^\star = \frac{z+m_1\ell|q|\varpi}{m+m_1}, \qquad w^\star = \frac{z-m\ell|q|\varpi}{m+m_1}$$

where $z=mv+m_1w$, $q=v-w$ and ϖ is a parameter vector on the sphere $\mathbb{S}^2$. The parameter ℓ is positive and such that $v^\star-w^\star=\ell|v-w|\varpi$. In particular, one sees from the representation (2.17) that $0<\ell\leq 1$. In this representation, one has

$$|v^\star|^2 = \frac{1}{(m+m_1)^2}\left(|z|^2+m_1^2\ell^2|q|^2+2\ell m_1|q||z|\cos\mu\right)$$

and

$$|w^\star|^2 = \frac{1}{(m+m_1)^2}\left(|z|^2+m^2\ell^2|q|^2-2\ell m|q||z|\cos\mu\right),$$

where μ is the angle between z and ϖ. One has then

$$m|v^\star|^2+m_1|w^\star|^2 = \frac{1}{m+m_1}\left(|z|^2+\ell^2 mm_1|q|^2\right). \tag{4.10}$$

One can check that

$$\begin{aligned}(m|v^\star|^2)(m_1|w^\star|^2) = \frac{mm_1}{(m+m_1)^4}\Big\{&\left[|z|^2+\ell^2mm_1|q|^2\right]^2-\left[|z|^2-\ell^2mm_1|q|^2\right]^2\cos^2\mu\\ &+\left[\ell(m_1-m)|z||q|+\left(|z|^2-\ell^2mm_1|q|^2\right)\cos\mu\right]^2\\ &-4\ell^2mm_1|z|^2|q|^2\cos^2\mu\Big\},\end{aligned}$$

i.e.

$$(m|v^\star|^2)(m_1|w^\star|^2) \geq \frac{mm_1}{(m+m_1)^4}\left[|z|^2+\ell^2mm_1|q|^2\right]^2(1-\cos^2\mu).$$

Therefore

$$\Delta(v^\star,w^\star) \geq \frac{mm_1}{(m+m_1)^2}\sin^2\mu.$$

We obtain then an estimate similar to the one obtained in [30]. Moreover, it is easy to see from (4.10) that

$$m|v^\star|^2+m_1|w^\star|^2 \geq \ell^2\left(m|v|^2+m_1|w|^2\right)$$

and, arguing as in [30], there exists some constant $\eta > 0$ such that

$$\mathfrak{n}_e[\Psi](v,w) \geq \eta \sin^2\mu \left(m|v|^2 + m_1|w|^2\right)^2 \Psi''\left(m|v|^2 + m_1|w|^2\right). \tag{4.11}$$

This allows to prove the following:

Lemma 4.1 (Povzner-like estimates for $\mathcal{L}$) *Let $\Psi(x) = x^r$, $r > 1$. Then, there exist positive constants k_r and A_r such that*

$$|v-w|\mathcal{J}_e[|\cdot|^{2r}](v,w) \leq A_r\Big(|v||w|^{2r} + |v|^{2r}|w|\Big) + \frac{m_1^r}{m^r}|v-w||w|^{2r} - k_r\Big(|v|^{2r+1} + |w|^{2r+1}\Big),$$

for any $v, w \in \mathbb{R}^3$.

Proof Bearing in mind that $\mathcal{J}_e[|\cdot|^{2r}](v,w)$ is provided by (4.7) and (4.8), first of all, since $\Psi(m_1|w^\star|^2) \geq 0$, we note that

$$\Psi(m|v^\star|^2) - \Psi(m|v|^2) \leq \mathfrak{q}_e[\Psi](v,w) + \Psi\left(m_1|w|^2\right) = \mathfrak{q}_e[\Psi](v,w) + m_1^r|w|^{2r}.$$

Then, integrating (4.9) and (4.11) with respect to the angle $\mathrm{n} \in \mathbb{S}^2$, one obtains, as in [30, Lemma 3.3.] and [30, Lemma 3.4], that there are A_r and $k_r > 0$ such that, for any $v, w \in \mathbb{R}^3$:

$$|v-w|\frac{1}{2\pi m^r}\int_{\mathbb{S}^2} \mathfrak{q}_e(\Psi)(v,w)|\hat{q}\cdot \mathrm{n}|\,\mathrm{dn} \leq A_r\Big(|v||w|^{2r} + |v|^{2r}|w|\Big) - k_r\Big(|v|^{2r+1} + |w|^{2r+1}\Big),$$

and this concludes the proof. □

The above Lemma (restoring $m = 1$) together with the known estimates for $Q_r(t)$ allow to formulate the following

Proposition 4.2 (Propagation of moments) *Let $f(t)$ be a solution to (1.1) with unit mass. For any $r \geq 1$, let*

$$\mathbb{Y}_r(t) = \int_{\mathbb{R}^3} f(v,t)|v|^{2r}\,\mathrm{d}v, \qquad t \geq 0.$$

Then, there are positive constants $\mathbf{A}_r$, $\mathbf{K}_r$ and $\mathbf{C}_r$ that depend only on $r, \alpha, \beta, \tau, \lambda$ and the moments of $\mathbf{F}_1$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{Y}_r(t) \leq \mathbf{C}_r + \mathbf{A}_r\mathbb{Y}_r(t) - \mathbf{K}_r\mathbb{Y}_r^{1+1/2r}(t), \qquad \forall t \geq 0.$$

As a consequence, if $\mathbb{Y}_r(0) < \infty$, then $\sup_{t\geq 0}\mathbb{Y}_r(t) < \infty$.

Proof Recall that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{Y}_r(t) = \tau Q_r(t) + L_r(t)$, where

$$Q_r(t) = \int_{\mathbb{R}^3} \mathcal{Q}(f,f)(v,t)|v|^{2r}\,\mathrm{d}v, \qquad L_r(t) = \int_{\mathbb{R}^3} \mathcal{L}(f)(v,t)|v|^{2r}\,\mathrm{d}v.$$

According to [30, Lemma 3.4], there exist $\widetilde{A}_r > 0$ and $\widetilde{k}_r > 0$ such that

$$Q_r(t) \leq \widetilde{A}_r\mathbb{Y}_{1/2}(t)\mathbb{Y}_r(t) - \widetilde{k}_r\mathbb{Y}_{r+1/2}(t), \quad t \geq 0.$$

Now, from Lemma 4.1

$$\lambda L_r(t) \leq A_r M_r \mathbb{Y}_{1/2}(t) + A_r M_{1/2} \mathbb{Y}_r(t) + m_1^r \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-w||w|^{2r} f(v,t)\mathbf{F}_1(w)\,\mathrm{d}w\,\mathrm{d}v$$
$$- k_r \mathbb{Y}_{r+1/2}(t) - k_r M_{r+1/2},$$

where $M_s = \int_{\mathbb{R}^3} |w|^{2s}\mathbf{F}_1(w)\,\mathrm{d}w,\ s \geq 1$. One has

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-w||w|^{2r} f(v,t)\mathbf{F}_1(w)\,\mathrm{d}w\,\mathrm{d}v \leq M_r \mathbb{Y}_{1/2}(t) + M_{r+1/2}$$

and, denoting $c_{1/2} := \sup_{t\geq 0} \mathbb{Y}_{1/2}(t) < \infty$, one has

$$L_r(t) \leq \mathbf{C}_r + \frac{A_r M_{1/2}}{\lambda}\mathbb{Y}_r(t) - \frac{k_r}{\lambda}\mathbb{Y}_{r+1/2}(t)$$

where $\mathbf{C}_r = (c_{1/2} A_r M_r + c_{1/2} m_1^r M_r + m_1^r M_{r+1/2})/\lambda$ is a positive constant depending only on $\alpha, \beta, \lambda, r \geq 1$ and the moments of $\mathbf{F}_1$. Gathering all these estimates leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{Y}_r(t) \leq \mathbf{C}_r + \mathbf{A}_r \mathbb{Y}_r(t) - \mathbf{K}_r \mathbb{Y}_{r+1/2}(t)$$

where $\mathbf{A}_r = \tau \widetilde{A}_r c_{1/2} + \frac{1}{\lambda} A_r M_{1/2} > 0$ and $\mathbf{K}_r = \tau \widetilde{k}_r + \frac{k_r}{\lambda} > 0$. Now, thanks to the mass conservation and Hölder's inequality, one gets $\mathbb{Y}_{r+1/2}(t) \geq \mathbb{Y}_r^{1+1/2r}(t)$ which leads to the desired result. □

Remark 4.3 We see from the definition of the positive constants $\mathbf{A}_r$, $\mathbf{C}_r$ and $\mathbf{K}_r$ that the above Proposition still holds true whenever $\tau = 0$ (i.e. for the linear problem).

4.4 Propagation of Lebesgue Norms

Let us consider now an initial condition $f_0 \in L^1_2 \cap L^p$ for some $1 < p < \infty$. We compute the time derivative of the L^p norm of the solution $f(v,t)$ to (1.1):

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f^p(v,t)\,\mathrm{d}v = \tau \int_{\mathbb{R}^3} \mathcal{Q}^+(f,f) f^{p-1}\,\mathrm{d}v - \tau \int_{\mathbb{R}^3} f^{p-1}\mathcal{Q}^-(f,f)\,\mathrm{d}v$$
$$+ \int_{\mathbb{R}^3} \mathcal{L}^+(f) f^{p-1}\,\mathrm{d}v - \int_{\mathbb{R}^3} \mathcal{L}^-(f) f^{p-1}\,\mathrm{d}v.$$

Using the fact that $\int_{\mathbb{R}^3} f^{p-1}\mathcal{Q}^-(f,f)\,\mathrm{d}v \geq 0$ and $\mathcal{L}^-(f)(v) = \nu(v) f(v)$ where the collision frequency $\nu(v)$ is given by

$$\nu(v) = \frac{1}{2\pi\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v-w)\cdot \mathrm{n}|\mathbf{F}_1(w)\,\mathrm{d}w\,\mathrm{dn},$$

we obtain the estimate:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f^p(v,t)\,\mathrm{d}v \leq \tau \int_{\mathbb{R}^3} \mathcal{Q}^+(f,f) f^{p-1}\,\mathrm{d}v + \int_{\mathbb{R}^3} \mathcal{L}^+(f) f^{p-1}\,\mathrm{d}v - \int_{\mathbb{R}^3} \nu(v) f^p(v,t)\,\mathrm{d}v.$$

Using the lower bound (2.15), we get

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p \le \tau \int_{\mathbb{R}^3} \mathcal{Q}^+(f,f) f^{p-1}\,\mathrm{d}v + \int_{\mathbb{R}^3} \mathcal{L}^+(f) f^{p-1}\,\mathrm{d}v - \chi \|f\|_{L^p_{1/p}}^p.$$

Proposition 3.1 and the conservation of mass imply that, for any $\delta > 0$, there is $\theta > 0$ and some C_δ such that

$$\int_{\mathbb{R}^3} \mathcal{Q}^+(f,f) f^{p-1}(v,t)\,\mathrm{d}v \le C_\delta \|f(t)\|_{L^p}^{p(1-\theta)} + \delta \|f(t)\|_{L^1_2} \|f(t)\|_{L^p_{1/p}}^p.$$

Moreover, Proposition 3.2 implies that, for any $\delta > 0$,

$$\int_{\mathbb{R}^3} \mathcal{L}^+(f) f^{p-1}\,\mathrm{d}v \le C_1 \|f(t)\|_{L^p}^{p-1} + C_2\delta \left(\|f(t)\|_{L^p_{1/p}}^p + \|f(t)\|_{L^1_2} \|f(t)\|_{L^p_{1/p}}^{p-1} \right),$$

for some constants $C_1, C_2 > 0$ that depend only on $p, \delta, \eta, \alpha, e$ and the norms of $\mathbf{F}_1$ in the spaces involved in (3.2). Recall that there is some M_2 such that

$$\sup_{t\ge 0} \|f(t)\|_{L^1_2} = 1 + \sup_{t \ge 0} \int_{\mathbb{R}^3} |v|^2 f(v,t)\,\mathrm{d}v \le M_2 < \infty.$$

Now, using Young's inequality, $xy^{p-1} \le \frac{1}{p}x^p + \frac{p-1}{p}y^p$, for any $x, y \ge 0$, we have

$$\int_{\mathbb{R}^3} \mathcal{L}^+(f) f^{p-1}\,\mathrm{d}v \le C_1 \|f(t)\|_{L^p}^{p-1} + C_3\delta \left(\|f(t)\|_{L^p_{1/p}}^p + M_2^p \right)$$

for some constant $C_3 > 0$. Collecting all the bounds above, we get the estimate

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p &\le \tau C_\delta \|f(t)\|_{L^p}^{p(1-\theta)} + \delta\tau M_2 \|f(t)\|_{L^p_{1/p}}^p + C_1 \|f(t)\|_{L^p}^{p-1} \\
&\quad + C_3\delta \left(\|f(t)\|_{L^p_{1/p}}^p + M_2^p \right) - \chi \|f(t)\|_{L^p_{1/p}}^p \\
&\le \tau C_\delta \|f(t)\|_{L^p}^{p(1-\theta)} + \left(\delta(\tau M_2 + C_3) - \frac{\chi}{2} \right) \|f(t)\|_{L^p_{1/p}}^p \\
&\quad + C_1 \|f(t)\|_{L^p}^{p-1} + C_3\delta M_2^p - \frac{\chi}{2} \|f(t)\|_{L^p}^p,
\end{aligned}$$

since $\|\cdot\|_{L^p_{1/p}} \ge \|\cdot\|_{L^p}$. Choosing now $\bar{\delta}$ such that $\bar{\delta}(\tau M_2 + C_3) < \chi/2$, we get the existence of positive constants C_4, C_5 and C_6 such that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p \le C_4 \|f(t)\|_{L^p}^{p(1-\theta)} + C_5 \|f(t)\|_{L^p}^{p-1} - C_6 \|f(t)\|_{L^p}^p + C_3 M_2^p \bar{\delta}.$$

It is not difficult to get then that $\sup_{t\ge 0} \|f(t)\|_{L^p} < \infty$. This can be summarized in the following

Proposition 4.4 (Propagation of L^p-norms) *Let $p \in (1,\infty)$ and $f_0 \in L^1_2 \cap L^p$ with unit mass. Then, the solution $f(t)$ to (1.1) satisfies the following uniform bounds*

$$\sup_{t\ge 0} \left(\|f(t)\|_{L^1_2} + \|f(t)\|_{L^p} \right) < \infty.$$

Remark 4.5 Notice that the fact that $\mathbf{F}_1$ is of finite entropy (see Assumption 2.1) has been used here above, via the lower bound (2.15), in order to control from below L^p norms involving the loss operator $\mathcal{L}^-$. Whenever $\tau > 0$, it is possible then to replace such estimates involving $\mathcal{L}^-$ by others that involve $\mathcal{Q}^-$. Notice also that, whenever $\tau = 0$ (i.e. in the linear case), only the above constant C_4 vanishes and we still have $\sup_{t\geq 0}\|f(t)\|_{L^p} < \infty$.

As a corollary, we deduce as in [37, Sect. 3.4], see also [25], the following non-concentration result:

Proposition 4.6 (Uniform non-concentration) *Let f_0 be given with unit mass. Assume that there exists some $p \in (1,\infty)$ such that $f_0 \in L^1_2 \cap L^p$. Then, there exists some positive constant ν_0 such that*

$$\nu_0 \leq \int_{\mathbb{R}^3} |v - \mathbf{u}(t)|^2 f(v,t)\,\mathrm{d}v \leq 1/\nu_0, \quad \forall t \geq 0,$$

where $f(v,t)$ is the solution to (1.1) with $f(0) = f_0$ and $\mathbf{u}(t) = \int_{\mathbb{R}^3} v f(v,t)\,\mathrm{d}v$, $t \geq 0$.

Proof Let $f(t)$ be the solution to (1.1) with $f(0) = f_0$. From the above Proposition, there exists $C_p > 0$ such that $\sup_{t\geq 0}\|f(t)\|_{L^p} \leq C_p$, and Hölder's inequality implies that, for any $r > 0$,

$$\sup_{t\geq 0}\int_{\{|v-\mathbf{u}(t)|<r\}} f(v,t)\,\mathrm{d}v \leq C_p \left(\frac{4\pi}{3} r^3\right)^{\frac{p-1}{p}}.$$

Accordingly, there is some $r_0 > 0$ such that

$$\int_{\{|v-\mathbf{u}(t)|<r_0\}} f(v,t)\,\mathrm{d}v \leq \frac{1}{2}, \quad \forall t \geq 0.$$

Then, for any $t \geq 0$, recalling that $\int_{\mathbb{R}^3} f(v,t)\,\mathrm{d}v = 1$ for any $t \geq 0$,

$$\begin{aligned}\int_{\mathbb{R}^3} f(v,t)|v-\mathbf{u}(t)|^2\,\mathrm{d}v &\geq \int_{\{|v-\mathbf{u}(t)|\geq r_0\}} f(v,t)|v-\mathbf{u}(t)|^2\,\mathrm{d}v \geq r_0^2 \int_{\{|v-\mathbf{u}(t)|\geq r_0\}} f(v,t)\,\mathrm{d}v \\ &\geq r_0^2\left(1 - \int_{\{|v-\mathbf{u}(t)|<r_0\}} f(v,t)\,\mathrm{d}v\right) \geq \frac{r_0^2}{2}\end{aligned}$$

which concludes the proof. □

4.5 L^1-Stability

As in [37], in order to prove the strong continuity of the semi-group $(\mathcal{S}_t)_{t\geq 0}$ associated to (1.1), one has to provide an estimate of $\|f(t) - g(t)\|$ for two solutions $f(t)$ and $g(t)$ of (1.1) with initial conditions $f(0), g(0)$ in some subspace of L^1. This is the object of the following stability result, inspired by [37, Proposition 3.2] and [41, Proposition 3.4].

Proposition 4.7 (L^1-stability) *Let f_0, g_0 be two nonnegative functions of L^1_3 and let $f(t), g(t) \in \mathcal{C}(\mathbb{R}^+, L^1_2) \cap L^\infty(\mathbb{R}^+, L^1_3)$ be the associated solutions to (1.1). Then, there is $\Lambda > 0$ depending only on $\sup_{t\geq 0}\|f(t) + g(t)\|_{L^1_3}$ such that*

$$\|f(t) - g(t)\|_{L^1_2} \leq \|f_0 - g_0\|_{L^1_2} \exp(\Lambda t), \quad \forall t \geq 0.$$

Proof Let $h(t) = f(t) - g(t)$. Then, h satisfies the following equation:

$$\partial_t h(v,t) = \tau\{\mathcal{Q}(f,f) - \mathcal{Q}(g,g)\} + \mathcal{L}(h), \quad h(0) = f_0 - g_0. \tag{4.12}$$

As in [37, 41], the proof consists in multiplying (4.12) by $\psi(v,t) = \mathrm{sgn}(h(v,t))\langle v\rangle^2$ and integrating over $\mathbb{R}^3$. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |h(v,t)|\langle v\rangle^2\,\mathrm{d}v = I(t) + L(t)$$

where

$$I(t) = \tau\int_{\mathbb{R}^3}\{\mathcal{Q}(f,f) - \mathcal{Q}(g,g)\}\psi(v,t)\,\mathrm{d}v \quad \text{and} \quad L(t) = \int_{\mathbb{R}^3}\mathcal{L}(h)(v,t)\psi(v,t)\,\mathrm{d}v.$$

To estimate the integral $I(t)$ we resume the arguments of [41, Proposition 3.4] that we shall need again later. According to (2.10)

$$I(t) = \frac{\tau}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(f(v,t)f(w,t) - g(v,t)g(w,t))\,|q|\mathcal{A}_\zeta[\psi(t)](v,w)\,\mathrm{d}w\,\mathrm{d}v.$$

The change of variables $(v,w) \mapsto (w,v)$ implies that

$$I(t) = \frac{\tau}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(f(v,t) - g(v,t))\,(f(w,t) + g(w,t))\,|q|\mathcal{A}_\zeta[\psi(t)](v,w)\,\mathrm{d}w\,\mathrm{d}v.$$

Moreover, it is easily seen from the definition of ψ that

$$\begin{aligned}
&(f(v,t) - g(v,t))\,\mathcal{A}_\zeta[\psi(t)](v,w)\\
&\quad\le \frac{1}{4\pi}\,|f(v,t) - g(v,t)|\int_{\mathbb{S}^2}\left(\langle v'\rangle^2 + \langle w'\rangle^2 - \langle v\rangle^2 + \langle w\rangle^2\right)\mathrm{d}\sigma\\
&\quad\le 2\,|f(v,t) - g(v,t)|\,\langle w\rangle^2
\end{aligned}$$

where we used the fact that $\mathcal{A}_\zeta[\langle\cdot\rangle^2](v,w) = -\frac{1-\epsilon^2}{4}|q|^2 \le 0$. Therefore,

$$\begin{aligned}
I(t) &\le \tau\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|q|\,|f(v,t) - g(v,t)|\,(f(w,t) + g(w,t))\,\langle w\rangle^2\,\mathrm{d}v\,\mathrm{d}w\\
&\le \tau\int_{\mathbb{R}^3}|f(v,t) - g(v,t)|\,\langle v\rangle^2\,\mathrm{d}v\int_{\mathbb{R}^3}(f(w,t) + g(w,t))\,\langle w\rangle^3\,\mathrm{d}w
\end{aligned}$$

i.e.

$$I(t) \le \tau\left\|f(t) + g(t)\right\|_{L^1_3}\left\|f(t) - g(t)\right\|_{L^1_2}, \quad \forall t \ge 0. \tag{4.13}$$

On the other hand, recalling that $\mathcal{L}(h)(v,t) = \mathcal{L}^+(h)(v,t) - \nu(v)h(v,t)$, from formula (3.3) one has

$$\begin{aligned}
L(t) &= \int_{\mathbb{R}^3}\psi(v,t)\,\mathrm{d}v\int_{\mathbb{R}^3}h(w,t)k(v,w)\,\mathrm{d}w - \int_{\mathbb{R}^3}\nu(v)|h(v,t)|\langle v\rangle^2\,\mathrm{d}v\\
&\le \int_{\mathbb{R}^3}\langle v\rangle^2\,\mathrm{d}v\int_{\mathbb{R}^3}|h(w,t)|k(v,w)\,\mathrm{d}w - \int_{\mathbb{R}^3}\nu(v)|h(v,t)|\langle v\rangle^2\,\mathrm{d}v,
\end{aligned}$$

i.e. $L(t) \leq \int_{\mathbb{R}^3} \mathcal{L}(|h|)(v,t)\langle v\rangle^2\,\mathrm{d}v$. Now, since $\int_{\mathbb{R}^3} \mathcal{L}(|h|)(v,t)\,\mathrm{d}v = 0$ for any h, one gets that

$$L(t) \leq \int_{\mathbb{R}^3} \mathcal{L}(|h|)(v,t)|v|^2\,\mathrm{d}v.$$

Resuming the calculations performed in Sect. 4.2 (see (4.4)), one gets that

$$\begin{aligned}\int_{\mathbb{R}^3} \mathcal{L}(|h|)(v,t)|v|^2\,\mathrm{d}v &\leq -\frac{2\kappa(1-\kappa)}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-w|^3|h(v,t)|\mathbf{F}_1(w)\,\mathrm{d}v\,\mathrm{d}w \\ &\quad + \frac{2\kappa}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q|\langle q,-w\rangle|h(v,t)|\mathbf{F}_1(w)\,\mathrm{d}v\,\mathrm{d}w \\ &\leq \frac{2\kappa}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q|^2|w||h(v,t)|\mathbf{F}_1(w)\,\mathrm{d}v\,\mathrm{d}w.\end{aligned}$$

This leads to

$$L(t) \leq \frac{2\kappa}{\lambda}\left(2\int_{\mathbb{R}^3}|v|^2|h(v,t)|\,\mathrm{d}v\int_{\mathbb{R}^3}|w|\mathbf{F}_1(w)\,\mathrm{d}w + 2\int_{\mathbb{R}^3}|h(v,t)|\,\mathrm{d}v\int_{\mathbb{R}^3}|w|^3\mathbf{F}_1(w)\,\mathrm{d}w\right)$$

and, setting $c_+ = \frac{4\kappa}{\lambda}\max\{\int_{\mathbb{R}^3}|w|\mathbf{F}_1(w)\,\mathrm{d}w, \int_{\mathbb{R}^3}|w|^3\mathbf{F}_1(w)\,\mathrm{d}w\}$, we get

$$L(t) \leq c_+\int_{\mathbb{R}^3}|h(v,t)|\langle v\rangle^2\,\mathrm{d}v = c_+\|f(t)-g(t)\|_{L^1_2}.$$

Gathering (4.13) together with the latter estimate and denoting then $\Lambda = \tau\sup_{t\geq 0}\|f(t)+g(t)\|_{L^1_3} + c_+$, we get the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)-g(t)\|_{L^1_2} \leq \Lambda\|f(t)-g(t)\|_{L^1_2}, \quad t\geq 0$$

and the proof is achieved. □

4.6 Well-Posedness of the Cauchy Problem

We are in position to prove that the Boltzmann equation (1.1) admits a unique regular solution in the following sense:

Theorem 4.8 (Existence and uniqueness of solution to the Cauchy problem) *Take an initial datum $f_0 \in L^1_3$. Then, for all $T>0$, there exists a unique solution $f \in \mathcal{C}([0,T];L^1_2) \cap L^\infty(0,T;L^1_3)$ to the Boltzmann equation (1.1) such that $f(v,0) = f_0(v)$.*

Proof Let $T>0$ be fixed. The uniqueness in $\mathcal{C}([0,T];L^1_2)\cap L^\infty(0,T;L^1_3)$ trivially follows from Proposition 4.7. The proof of the existence is made in several steps, following the lines of [41, Sect. 3.3], see also [29, 40].

Step 1. Let us first consider an initial datum $f_0 \in L^1_4$, and define the "truncated" collision operators

$$\int_{\mathbb{R}^3} \mathcal{Q}_n(f,f)(v)\psi(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{1}_{\{|q|\leq n\}}|q| f(v) f(w)\mathcal{A}_\zeta[\psi](v,w)\,\mathrm{d}w\,\mathrm{d}v,$$
$$\int_{\mathbb{R}^3} \mathcal{L}_n(f)(v)\psi(v)\,\mathrm{d}v = \frac{1}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{1}_{\{|q|\leq n\}}|q| f(v)\mathbf{F}_1(w)\mathcal{J}_e[\psi](v,w)\,\mathrm{d}v\,\mathrm{d}w \tag{4.14}$$

for any regular test function ψ. The operators $\mathcal{Q}_n$ and $\mathcal{L}_n$ are bounded in any L^1_q, and they are Lipschitz in L^1_2 on any bounded subset of L^1_2. Therefore, following [1], we can use the Banach fixed point theorem to get the existence of a solution $0 \leq f_n \in \mathcal{C}([0,T];L^1_2) \cap L^\infty(0,T;L^1_4)$ to the Boltzmann equation $\partial_t f = \tau\mathcal{Q}_n(f,f) + \mathcal{L}_n(f)$. Thanks to the uniform propagation of moments in Proposition 4.2, there exists a constant $C_T > 0$ (that does not depend on n) such that

$$\sup_{[0,T]} \|f_n\|_{L^1_4} \leq C_T, \quad \forall n \in \mathbb{N}.$$

Step 2. Let us prove that the sequence $(f_n)_n$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1_2) \cap L^\infty(0,T;L^1_4)$. For any $m \geq n$, writing down the equation satisfied by $f_m - f_n$ and multiplying it by $\psi(v,t) = \mathrm{sgn}(f_m(v,t) - f_n(v,t))\langle v\rangle^2$ as in the proof of Proposition 4.7, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |f_n(v,t) - f_m(v,t)|\langle v\rangle^2\,\mathrm{d}v = I_{m,n}(t) + J_{m,n}(t)$$

where

$$I_{m,n}(t) = \tau\int_{\mathbb{R}^3}\{\mathcal{Q}_m(f_m,f_m) - \mathcal{Q}_n(f_n,f_n)\}\psi(v,t)\,\mathrm{d}v$$

and

$$J_{m,n}(t) = \int_{\mathbb{R}^3}\{\mathcal{L}_m(f_m)(v,t) - \mathcal{L}_n(f_n)(v,t)\}\psi(v,t)\,\mathrm{d}v.$$

We begin by estimating $I_{m,n}(t)$. It is easy to see that $I_{m,n}(t) = I^1_{m,n}(t) + I^2_{m,n}(t)$ where

$$I^1_{m,n}(t) = \frac{\tau}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(f_m(v,t)f_m(w,t) - f_n(v,t)f_n(w,t)\Big)B_m(q)\mathcal{A}_\zeta[\psi(t)](v,w)\,\mathrm{d}w\,\mathrm{d}v,$$

while

$$I^2_{m,n}(t) = \frac{\tau}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(B_m(q) - B_n(q))\,f_n(v,t)f_n(w,t)\mathcal{A}_\zeta[\psi(t)](v,w)\,\mathrm{d}w\,\mathrm{d}v,$$

where $B_n(q) = |q|\mathbf{1}_{\{|q|\leq n\}}$. Arguing as in the proof of (4.13), we get easily that

$$I^1_{m,n}(t) \leq \tau\left\|f_n(t) + f_m(t)\right\|_{L^1_3}\left\|f_n(t) - f_m(t)\right\|_{L^1_2}, \quad \forall t \geq 0.$$

The estimate of $I^2_{m,n}(t)$ is more involved. One observes first that

$$B_m(q) - B_n(q) = |q|\mathbf{1}_{\{n\leq|q|\leq m\}} \leq |q|\left(\mathbf{1}_{\{|v|\geq n/2\}} + \mathbf{1}_{\{|w|\geq n/2\}}\right).$$

As in the proof of Proposition 4.7, one has

$$\mathcal{A}_\zeta[\psi(t)](v,w) \leq \frac{1}{4\pi}\int_{\mathbb{S}^2}\left(\langle v'\rangle^2 + \langle w'\rangle^2 + \langle v\rangle^2 + \langle w\rangle^2\right)\mathrm{d}\sigma \leq 2\left(\langle v\rangle^2 + \langle w\rangle^2\right)$$

and, since $|q| \le \langle v\rangle\langle w\rangle$, one gets

$$I^2_{m,n}(t) \le \tau \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_n(v,t) f_n(w,t)|q| \left(\mathbf{1}_{\{|v|\ge n/2\}} + \mathbf{1}_{\{|w|\ge n/2\}}\right)\left(\langle v\rangle^2 + \langle w\rangle^2\right) \mathrm{d}w\,\mathrm{d}v$$
$$\le \tau \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_n(v,t) f_n(w,t)\langle v\rangle\langle w\rangle \left(\langle v\rangle^2 + \langle w\rangle^2\right)\left(\mathbf{1}_{\{\langle v\rangle\ge n/2\}} + \mathbf{1}_{\{\langle w\rangle\ge n/2\}}\right) \mathrm{d}w\,\mathrm{d}v.$$

It is not difficult to deduce then that

$$I^2_{m,n}(t) \le 4\tau \left(\int_{\mathbb{R}^3} f_n(v,t)\langle v\rangle^3\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^3} f_n(v,t)\langle v\rangle^3 \mathbf{1}_{\{\langle v\rangle\ge n/2\}}\,\mathrm{d}v\right).$$

Since $\sup_{[0,T]}\|f_n(t)\|_{L^1_4} \le C_T$ for any $n\in\mathbb{N}$, the latter integral is estimated as

$$\int_{\mathbb{R}^3} f_n(v,t)\langle v\rangle^3 \mathbf{1}_{\{\langle v\rangle\ge n/2\}}\,\mathrm{d}v \le \int_{\mathbb{R}^3} f_n(v,t)\langle v\rangle^4 \mathbf{1}_{\{\langle v\rangle\ge n/2\}}\frac{\mathrm{d}v}{\langle v\rangle} \le \frac{2C_T}{n}$$

and we get

$$I^2_{m,n}(t) \le 4\tau\left(\int_{\mathbb{R}^3} f_m(v,t)\langle v\rangle^3\,\mathrm{d}v\right)\frac{2C_T}{n} \le \frac{8C_T^2\tau}{n}, \quad \forall t\in[0,T],\ m\ge n.$$

Therefore,

$$I_{m,n}(t) \le 2\tau C_T \left\|f_n(t) - f_m(t)\right\|_{L^1_2} + \frac{8C_T^2\tau}{n}, \quad \forall t\in[0,T],\ m\ge n. \tag{4.15}$$

We proceed in the same way with $J_{m,n}(t)$. First, we notice that $J_{m,n}(t)$ splits as $J_{m,n}(t) = J^1_{m,n}(t) + J^2_{m,n}(t)$ with

$$J^1_{m,n}(t) = \frac{1}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B_m(q)\left[f_m(v,t) - f_n(v,t)\right]\mathbf{F}_1(w)\mathcal{J}_e\left[\psi(t)\right](v,w)\,\mathrm{d}v\,\mathrm{d}w$$

and

$$J^2_{m,n}(t) = \frac{1}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[B_m(q) - B_n(q)\right] f_n(v,t)\mathbf{F}_1(w)\mathcal{J}_e\left[\psi(t)\right](v,w)\,\mathrm{d}v\,\mathrm{d}w.$$

Arguing as in the proof of Proposition 4.7, we get

$$J^1_{m,n}(t) \le \int_{\mathbb{R}^3} \mathcal{L}_m(|f_n - f_m|)(v,t)\langle v\rangle^2\,\mathrm{d}v \le \frac{2\kappa}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q|^2|w||(f_n - f_m)(v,t)|\mathbf{F}_1(w)\,\mathrm{d}v\,\mathrm{d}w$$

and there exists a positive constant c_+ such that

$$J^1_{m,n}(t) \le c_+\left\|f_n(t) - f_m(t)\right\|_{L^1_2}, \quad \forall t\in[0,T].$$

Let us now estimate $J^2_{m,n}(t)$. As above,

$$J^2_{m,n}(t) = \frac{1}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q|\mathbf{1}_{\{n\le|q|\le m\}} f_n(v,t)\mathbf{F}_1(w)\mathcal{J}_e\left[\psi(t)\right](v,w)\,\mathrm{d}v\,\mathrm{d}w$$

and

$$\mathcal{J}_e\left[\psi(t)\right](v,w) \le \frac{1}{2\pi}\int_{\mathbb{S}^2} |\hat{q}\cdot \mathrm{n}|\left(\langle v^\star\rangle^2 + \langle v\rangle^2\right)\mathrm{d}\mathrm{n} = \mathcal{J}_e\left[\langle\cdot\rangle^2\right] + \frac{2}{2\pi}\int_{\mathbb{S}^2}|\hat{q}\cdot\mathrm{n}|\langle v\rangle^2\,\mathrm{d}\mathrm{n}.$$

Calculations already performed lead then to

$$\mathcal{J}_e\left[\psi(t)\right](v,w) \le -2\kappa\langle q, w\rangle + 2\langle v\rangle^2 \le 2\left(\langle v\rangle\langle w\rangle^2 + \langle v\rangle^2\right), \quad \forall v, w\in\mathbb{R}^3.$$

Finally,

$$\begin{aligned} J^2_{m,n}(t) &\le \frac{2}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |q|\mathbf{1}_{\{|q|\ge n\}} f_n(v,t)\mathbf{F}_1(w)\left(\langle v\rangle\langle w\rangle^2 + \langle v\rangle^2\right)\mathrm{d}w\,\mathrm{d}v \\ &\le \frac{2}{\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{1}_{\{|q|\ge n\}} f_n(v,t)\mathbf{F}_1(w)\left(\langle v\rangle^3\langle w\rangle + \langle v\rangle^2\langle w\rangle^3\right)\mathrm{d}w\,\mathrm{d}v. \end{aligned}$$

Now, arguing as we did for $I^2_{m,n}(t)$, there exists some constant $\tilde{C}_T$ that depends only on $\|\mathbf{F}_1\|_{L^1_4}$ and $\sup_n \sup_{[0,T]}\|f_n(t)\|_{L^1_4}$ such that

$$J^2_{m,n}(t) \le \frac{\tilde{C}_T}{n}, \quad \forall m\ge n,\ t\in[0,T].$$

Gathering all these estimates, we obtain the existence of constants $C_1(T)$ and $C_2(T)$ that do not depend on m, n such that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |f_n(v,t) - f_m(v,t)|\langle v\rangle^2\,\mathrm{d}v \\ &\quad\le C_1(T)\|f_n(t) - f_m(t)\|_{L^1_2} + \frac{C_2(T)}{n}, \quad \forall t\in[0,T],\ m\ge n. \end{aligned}$$

This ensures that $(f_n)_n$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1_2)$. Denoting by f its limit, we obtain that $f\in\mathcal{C}([0,T];L^1_2)\cap L^\infty(0,T;L^1_4)$ is a solution to the Boltzmann equation (1.1) (with the actual collision operators $\mathcal{Q}$ and $\mathcal{L}$).

Step 3. When the initial datum $f_0\in L^1_3$, we introduce the sequence of initial data $f_{0,j} := f_0\mathbf{1}_{\{|v|\le j\}}$. Since $f_{0,j}\in L^1_4$, we have the existence of a solution $f_j\in\mathcal{C}([0,T];L^1_2)\cap L^\infty(0,T;L^1_4)$ to the Boltzmann equation associated to the initial datum $f_{0,j}$. Moreover, there exists C_T such that $\sup_{[0,T]}\|f_j\|_{L^1_3}\le C_T$. We establish again that $(f_j)_j$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1_2)$ by using the L^1-stability in Proposition 4.7. □

5 Existence of Non-Trivial Stationary State

All the material of the previous sections allows us to state our main result:

Theorem 5.1 (Existence of stationary solutions) *For any distribution function* $\mathbf{F}_1(v)$ *satisfying Assumption* 2.1 *and any* $\tau\ge 0$, *there exists a nonnegative* $F\in L^1_2\cap L^p$, $p\in(1,\infty)$ *with unit mass and positive temperature such that* $\tau\mathcal{Q}(F,F)+\mathcal{L}(F)=0$.

Proof As already announced, the existence of stationary solution to (1.1) relies on the application of Lemma 2.4 to the evolution semi-group $(\mathcal{S}_t)_{t\ge0}$ governing (1.1). Namely, for

$f_0 \in L^1$, let $f(t) = \mathcal{S}_t f_0$ denote the unique solution to (1.1) with initial state $f(0) = f_0$. The continuity properties of the semi-group are proved by the study of the Cauchy problem, recalled in Sect. 4. Let us fix $p_o \in (1, \infty)$. On the Banach space $\mathcal{Y} = L^1_2$, thanks to the uniform bounds on the L^1_3 and L^{p_o} norms, the nonempty convex subset

$$\mathcal{Z} = \left\{ 0 \le f \in \mathcal{Y}, \ \int_{\mathbb{R}^3} f \, \mathrm{d}v = 1 \text{ and } \|f\|_{L^1_3} + \|f\|_{L^{p_o}} \le M \right\}$$

is stable by the semi-group provided M is big enough. This set is weakly compact in $\mathcal{Y}$ by Dunford-Pettis Theorem, and the continuity of $\mathcal{S}_t$ for all $t \ge 0$ on $\mathcal{Z}$ follows from Proposition 4.7. Then, Lemma 2.4 shows that there exists a nonnegative stationary solution to (1.1) in $L^1_3 \cap L^{p_o}$ with unit mass. In fact, the uniform in time L^p bounds also imply the boundedness of F in L^p for all $p \in (1, \infty)$. □

As a corollary of Theorem 5.1, choosing $\tau = 0$ allows us to prove the existence of a steady state to the *linear inelastic scattering operator* $\mathcal{L}$ when the distribution function of the background is not a Maxwellian, generalizing the result of [34, 36, 44].

Corollary 5.2 *Let* $\mathbf{F}_1$ *satisfy Assumption* 2.1. *Then, the linear inelastic scattering operator* $\mathcal{L}$ *defined by* (2.13) *admits a unique nonnegative steady state* $F \in L^1_2 \cap L^p$, $p \in (1, \infty)$, *with unit mass and positive temperature.*

Proof The existence of a nonnegative equilibrium solution $F \in L^1_2$ is a direct application of Theorem 5.1 with $\tau = 0$. A simple application of Krein-Rutman Theorem implies the uniqueness of the stationary solution F within the range of nonnegative distributions with unit mass. □

Remark 5.3 (H-Theorem and Trend Towards Equilibrium) As in [34], it is possible to prove a linear version of the classical H-Theorem for the linear inelastic Boltzmann equation (1.1) with $\tau = 0$:

$$\partial_t f = \mathcal{L}(f), \qquad f(t=0) = f_0 \in L^1. \tag{5.1}$$

Namely, for any *convex* $\mathcal{C}^1$ function $\Phi : \mathbb{R}^+ \to \mathbb{R}$, let

$$H_\Phi(f|F) = \int_{\mathbb{R}^3} F(v) \Phi\left(\frac{f(v)}{F(v)}\right) \mathrm{d}v, \quad f \in L^1.$$

Arguing as in [34], it is easy to prove that, if the initial state f_0 has unique mass and finite entropy $H_\Phi(f_0|F) < \infty$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} H_\Phi(f(t)|F) \le 0 \quad (t \ge 0) \tag{5.2}$$

where $f(t)$ stands for the (unique) solution to (5.1). Moreover, still arguing as in [34], one proves that if moreover $\int_{\mathbb{R}^3} (1 + v^2 + |\log f_0(v)|) f_0(v) \, \mathrm{d}v < \infty$, then

$$\lim_{t \to \infty} \int_{\mathbb{R}^3} |f(v,t) - F(v)| \, \mathrm{d}v = 0.$$

6 Regularity of the Steady State

In this final section, our aim is to establish the existence of some smooth stationary solution to (1.1). Namely, adopting the strategy of [37, Sect. 4.1], we prove

Theorem 6.1 (Regularity of stationary solutions) *There exists a stationary solution F to the Boltzmann equation*

$$\tau \mathcal{Q}(F,F) + \mathcal{L}(F) = 0$$

that belongs to $C^\infty(\mathbb{R}^3)$.

We shall follow the same lines of [43, Theorem 5.5] and [37, Sect. 3.6], from which we deduce the exponential decay in time of singularities and thus the smoothness of stationary solutions. This proof needs the following ingredients:

i) The stability result already proved in Proposition 4.7.
ii) An estimate on the Duhamel representation [37, Proposition 3.4] of the solution to (1.1) (see Proposition 6.2).
iii) A result of propagation of Sobolev norms (see Proposition 6.3).

Let us first extend the regularity estimate of [37, Proposition 3.4] to our situation. Recall that (see Sect. 4.1), for $f_0 \in L^1_3$, the unique solution $f(t)$ to (4.1) is given by the following Duhamel representation:

$$f(v,t) = f_0(v)G(v,0,t) + \int_0^t \left(\tau \mathcal{Q}^+(f,f) + \mathcal{L}^+(f)\right)(v,s)G(v,s,t)\,\mathrm{d}s \tag{6.1}$$

where we set

$$G(v,s,t) = \exp\left(-\int_s^t \Sigma(f)(v,r)\,\mathrm{d}r\right) \quad 0 \le s \le t, \;\; v \in \mathbb{R}^3.$$

Proposition 6.2 *There are some positive constants C_{Duh}, K such that for any $k \in \mathbb{N}$ and $\eta \ge 0$ we have*

$$\|f_0(\cdot)G(\cdot,0,t)\|_{H^{k+1}_\eta} \le C_{\mathrm{Duh}} \mathrm{e}^{-Kt} \|f_0\|_{H^{k+1}_{\eta+1}} \left(\sup_{0\le r\le t} \|f(\cdot,r)\|^2_{H^k_{\eta+3}} + \sup_{0\le r\le t} \|f(\cdot,r)\|^{k+3}_{H^k_{\eta+3}}\right) \tag{6.2}$$

and

$$\left\|\int_0^t G(\cdot,s,t)\left(\tau \mathcal{Q}^+(f,f) + \mathcal{L}^+(f)\right)(\cdot,s)\,\mathrm{d}s\right\|_{H^{k+1}_\eta} \le C_{\mathrm{Duh}} \left(\sup_{0\le r\le t} \|f(\cdot,r)\|^2_{H^k_{\eta+3}} + \sup_{0\le r\le t} \|f(\cdot,r)\|^{k+3}_{H^k_{\eta+3}}\right). \tag{6.3}$$

Proof The proof is quite similar to [43, Proposition 5.2]. Here, for simplicity we have done it for natural k, although it is simple to generalize it to $k > 0$ by interpolation. Precisely, for any $f \in L^1$ define

$$L(f)(v) = \int_{\mathbb{R}^3} |v-w| f(w)\,\mathrm{d}w.$$

It is clear that

$$\Sigma(f)(v) = L(\tau f + \mathbf{F}_1)(v), \quad \forall f \in L^1.$$

Now, according to [27, Lemma 4.3], for any given $k \geq 0$ and any $\delta > 3/2$, the linear operator

$$L : H^k_\delta \longrightarrow W^{k+1,\infty}_{-1}$$

is bounded, i.e. for any $\delta > 3/2$ and any $k \geq 0$, there exists $C_{k,\delta}$ such that

$$\|L(g)\|_{W^{k+1,\infty}_{-1}} \leq C_{k,\delta}\|g\|_{H^k_\delta}, \quad \forall g \in H^k_\delta.$$

Let us fix now $k \in \mathbb{N}$ and $\delta > 3/2$. Since $\mathbf{F}_1 \in H^k_\delta$ due to Assumption 2.1, one deduces that

$$\|\Sigma(f)\|_{W^{k+1,\infty}_{-1}} \leq C\|f + \mathbf{F}_1\|_{H^k_\delta}, \quad \forall f \in H^k_\delta$$

where, as in the rest of the proof, we shall denote any positive constant independent of f and possibly dependent on $\mathbf{F}_1$ by C. Setting

$$F(v,s,t) = \int_s^t \Sigma(f)(v,r)\,\mathrm{d}r,$$

one sees that

$$\|F(\cdot,s,t)\|_{W^{k+1,\infty}_{-1}} \leq C\sqrt{t-s}\left(\int_s^t \|f(\cdot,r)\|^2_{H^k_\delta}\,\mathrm{d}r\right)^{1/2} + C(t-s)\|\mathbf{F}_1\|_{H^k_\delta}, \quad 0 \leq s \leq t.$$

Now, since $L(g) \geq 0$ for any $g \geq 0$, according to Assumption 2.1 and (2.15), we see that there exists some constant $\chi > 0$ such that

$$\Sigma(f)(v) \geq L(\mathbf{F}_1)(v) \geq \chi, \quad \forall f \in L^1,\ f \geq 0,\ \forall v \in \mathbb{R}^3.$$

By taking the successive derivatives of $G(v,s,t) = \exp(-F(v,s,t))$, one gets as in [43, Proposition 5.2]

$$\begin{aligned}\|G(\cdot,s,t)\|_{W^{k+1,\infty}_{-1}} &\leq C\mathrm{e}^{-\chi(t-s)}\left[\sqrt{t-s}\left(\int_s^t \|f(\cdot,r)\|^2_{H^k_\delta}\,\mathrm{d}r\right)^{(k+1)/2} + (t-s)\|\mathbf{F}_1\|_{H^k_\delta} + 1\right]\\ &\leq C\mathrm{e}^{-K(t-s)}\left(1 + \sup_{s\leq r\leq t}\|f(\cdot,r)\|^{k+1}_{H^k_\delta}\right),\end{aligned} \tag{6.4}$$

for some $0 < K < \chi$. Then, we shall use the following estimate [43, Lemma 5.3] that allows to exchange a time integral and a Sobolev norm:

$$\left\|\int_0^t Z(\cdot,s)\,\mathrm{d}s\right\|_{H^r_\ell} \leq \frac{1}{\sqrt{\lambda}}\left(\int_0^t \mathrm{e}^{\lambda(t-s)}\|Z(\cdot,s)\|^2_{H^r_\ell}\,\mathrm{d}s\right)^{1/2}, \quad \forall \lambda > 0,\ \forall \ell, r \in \mathbb{R}.$$

As a consequence we have for any $k \geq 0$,

$$\left\|\int_0^t \big(\tau\mathcal{Q}^+(f,f) + \mathcal{L}^+(f)\big)(\cdot,s)G(\cdot,s,t)\,\mathrm{d}s\right\|_{H^{k+1}_\eta}$$

$$\leq C\left[\int_0^t \mathrm{e}^{K(t-s)}\left(\left\|\tau\mathcal{Q}^+(f,f)(\cdot,s)\right\|^2_{H^{k+1}_{\eta+1}} + \left\|\mathcal{L}^+(f)(\cdot,s)\right\|^2_{H^{k+1}_{\eta+1}}\right)\right.$$
$$\left.\times \|G(\cdot,s,t)\|^2_{W^{k+1,\infty}_{-1}}\,\mathrm{d}s\right]^{1/2}.$$

Recall now the so-called Bouchut-Desvillettes-Lu regularity result in Propositions 3.1 and 3.2:

$$\|\mathcal{Q}^+(f,f)\|_{H^{k+1}_{\eta+1}} \leq C\Big[\|f\|^2_{H^k_{\eta+3}} + \|f\|^2_{L^1_{\eta+3}}\Big]$$

and

$$\left\|\mathcal{L}^+(f)\right\|_{H^{k+1}_{\eta+1}} \leq C\left[\|\mathbf{F}_1\|_{H^k_{\eta+3}}\|f\|_{H^k_{\eta+3}} + \|\mathbf{F}_1\|_{L^1_{\eta+3}}\|f\|_{L^1_{\eta+3}}\right].$$

Arguing now as in [43, Proposition 5.2] and using the estimate (6.4) with the choice $\delta = \eta + 3$, we get

$$\left\|\int_0^t \big(\tau\mathcal{Q}^+(f,f) + \mathcal{L}^+(f)\big)(\cdot,s)G(\cdot,s,t)\,\mathrm{d}s\right\|_{H^{k+1}_\eta}$$
$$\leq C\left[\int_0^t \mathrm{e}^{K(t-s)}\|f(\cdot,s)\|^4_{H^k_{\eta+3}}\,\mathrm{e}^{-2K(t-s)}\left(1+\sup_{s\leq r\leq t}\|f(\cdot,r)\|^{k+1}_{H^k_{\eta+3}}\right)^2\mathrm{d}s\right]^{1/2}$$
$$\leq C\max\left(\sup_{0\leq r\leq t}\|f(\cdot,r)\|^2_{H^k_{\eta+3}},\ \sup_{0\leq r\leq t}\|f(\cdot,r)\|^{k+3}_{H^k_{\eta+3}}\right)$$

which proves (6.3). The proof of (6.2) is similar. □

A direct consequence of the previous result together with the uniform L^2 bounds is the uniform in time propagation of Sobolev norms. The proof is carried on exactly as in [37, Proposition 3.5].

Proposition 6.3 *Let* $\mathbf{F}_1$ *satisfy Assumption* 2.1. *Let* $f_0 \in L^1_2$, $f_0 \geq 0$ *with unit mass and let* f *be the unique solution of the Boltzmann equation* (1.1) *in* $\mathcal{C}(\mathbb{R}^+; L^1_2) \cap L^\infty(\mathbb{R}^+; L^1_3)$ *associated with* f_0. *Then, for all* $s > 0$ *and* $\eta \geq 1$, *there exists* $w(s) > 0$ *such that*

$$f_0 \in H^s_{\eta+w} \quad \Longrightarrow \quad \sup_{t\geq 0}\|f(\cdot,t)\|_{H^s_\eta} < +\infty.$$

The previous ingredients allow to proof the following theorem, see [43, Theorem 5.5] for the proof.

Theorem 6.4 (Exponential decay of singularities) *Let* $f_0 \in L^1_2 \cap L^2$ *with unit mass and let* f *be the unique solution of the Boltzmann equation* (1.1) *in* $\mathcal{C}(\mathbb{R}^+; L^1_2) \cap L^\infty(\mathbb{R}^+; L^1_3)$ *associated with* f_0. *Let* $\mathbf{F}_1$ *satisfy Assumption* 2.1. *Let* $s \geq 0$, $q \geq 0$ *be arbitrarily large. Then* f *splits into the sum of a regular and a singular part* $f = f_R + f_S$ *where*

$$\begin{cases} \sup_{t\geq 0}\|f_R(t)\|_{H^s_q\cap L^1_2} < +\infty, \quad f_R \geq 0 \\ \exists\lambda > 0: \quad \|f_S(t)\|_{L^1_2} = O\big(\mathrm{e}^{-\lambda t}\big). \end{cases}$$

Proof The proof is easily adapted from [37, Theorem 3.6] since the L^1-stability result (Proposition 4.7), the Duhamel representation (Proposition 6.2), the uniform propagation of Sobolev norms (Proposition 6.3) allow to adapt directly [43, Theorem 5.5]. □

Finally, Theorem 6.4 allows to prove the main Theorem 6.1.

Acknowledgements JAC acknowledges the support from the project MTM2008-06349-C03-03 DGI-MCI (Spain). JAC and MB acknowledge partial support of the Acc. Integ. program HI2006-0111. MB acknowledges support also from MIUR (Project "Non-conservative binary interactions in various types of kinetic models"), from GNFM–INdAM, and from the University of Parma. JAC acknowledges IPAM–UCLA where part of this work was done. Finally, BL wishes to thank Clément Mouhot for fruitful discussions.

References

1. Arkeryd, L.: On the Boltzmann equation. Arch. Ration. Mech. Anal. **34**, 1–34 (1972)
2. Arkeryd, L.: L^∞-estimates for the spatially homogeneous Boltzmann equation. J. Stat. Phys. **31**, 347–361 (1983)
3. Arlotti, L., Lods, B.: Integral representation of the linear Boltzmann operator for granular gas dynamics with applications. J. Stat. Phys. **129**, 517–536 (2007)
4. Bagland, V., Laurençot, P.: Self-similar solutions to the Oort-Hulst-Safronov coagulation equation. SIAM J. Math. Anal. **39**, 345–378 (2007)
5. Ben-Naim, E., Krapivski, P.: Multiscaling in inelastic collisions. Phys. Rev. E **61**, R5–R8 (2000)
6. Ben-Naim, E., Krapivski, P.: Nontrivial velocity distributions in inelastic gases. J. Phys. A **35**, L147–L152 (2002)
7. Biben, Th., Martin, Ph.A., Piasecki, J.: Stationary state of thermostated inelastic hard spheres. Physica A **310**, 309–324 (2002)
8. Bisi, M., Spiga, G.: Fluid-dynamic equations for granular particles in a host-medium. J. Math. Phys. **46**, 113301 (2005)
9. Bisi, M., Spiga, G.: Dilute granular flows in a medium. Transp. Theory Stat. Phys. **36**, 79–105 (2007)
10. Bisi, M., Carrillo, J.A., Toscani, G.: Contractive Metrics for a Boltzmann equation for granular gases: Diffusive equilibria. J. Stat. Phys. **118**, 301–331 (2005)
11. Bisi, M., Carrillo, J.A., Toscani, G.: Decay rates in probability metrics towards homogeneous cooling states for the inelastic Maxwell model. J. Stat. Phys. **124**, 625–653 (2006)
12. Bizon, C., Shattuck, M.D., Swift, J.B., Swinney, H.L.: Transport coefficients for granular media from molecular dynamics simulations. Phys. Rev. E **60**, 4340–4351 (1999)
13. Bobylev, A.V.: Fourier transform method in the theory of the Boltzmann equation for Maxwellian molecules. Dokl. Akad. Nauk USSR **225**, 1041–1044 (1975)
14. Bobylev, A.V., Cercignani, C.: Moment equations for a granular material in a thermal bath. J. Stat. Phys. **106**, 547–567 (2002)
15. Bobylev, A.V., Cercignani, C.: Self-similar asymptotics for the Boltzmann equation with inelastic and elastic interactions. J. Stat. Phys. **110**, 333–375 (2003)
16. Bobylev, A.V., Gamba, I.: Boltzmann equations for mixtures of Maxwell gases: exact solutions and power like tails. J. Stat. Phys. **124**, 497–516 (2006)
17. Bobylev, A.V., Carrillo, J.A., Gamba, I.: On some properties of kinetic and hydrodynamic equations for inelastic interactions. J. Stat. Phys. **98**, 743–773 (2000); Erratum on: J. Stat. Phys. **103**, 1137–1138 (2001)
18. Bobylev, A.V., Gamba, I., Panferov, V.: Moment inequalities and high-energy tails for the Boltzmann equations with inelastic interactions. J. Stat. Phys. **116**, 1651–1682 (2004)
19. Bobylev, A.V., Cercignani, C., Gamba, I.: On the self-similar asymptotics for generalized non-linear kinetic Maxwell models. Commun. Math. Phys. (2008, to appear)
20. Bouchut, F., Desvillettes, L.: A proof of the smoothing properties of the positive part of Boltzmann's kernel. Rev. Mat. Iberoam. **14**, 47–61 (1998)
21. Brilliantov, N.V., Pöschel, T.: Kinetic Theory of Granular Gases. Oxford University Press, London (2004)
22. Carlen, E.A., Carrillo, J.A., Carvalho, M.C.: Strong convergence towards homogeneous cooling states for dissipative Maxwell models. Ann. Inst. Henri Poincaré Anal. Non Linéaire (2008, to appear)
23. Carrillo, J.A., Toscani, G.: Contractive probability metrics and asymptotic behavior of dissipative kinetic equations. Riv. Mat. Univ. Parma (7) **6**, 75–198 (2007)

24. Carrillo, J.A., Cercignani, C., Gamba, I.: Steady states of a Boltzmann equation for driven granular media. Phys. Rev. E **62**, 7700–7707 (2000)
25. Carrillo, J.A., Di Francesco, M., Toscani, G.: Intermediate asymptotics beyond homogeneity and self-similarity: long time behavior for $u_t = \Delta\phi(u)$. Arch. Ration. Mech. Anal. **180**, 127–149 (2006)
26. Desvillettes, L.: About the use of the Fourier transform for the Boltzmann equation. Riv. Mat. Univ. Parma (7) **2**, 1–99 (2003)
27. Duduchava, R., Kirsch, R., Rjasanow, S.: On estimates of the Boltzmann collision operator with cut-off. J. Math. Fluid Mech. **8**, 242–266 (2006)
28. Escobedo, M., Mischler, S., Rodriguez Ricard, M.: On self-similarity and stationary problem for fragmentation and coagulation models. Ann. Inst. Henri Poincaré Anal. Non Linéaire **22**, 99–125 (2005)
29. Fournier, N., Mischler, S.: A spatially homogeneous Boltzmann equation for elastic, inelastic, and coalescing collisions. J. Math. Pures Appl. **84**, 1173–1234 (2005)
30. Gamba, I., Panferov, V., Villani, C.: On the Boltzmann equation for diffusively excited granular media. Commun. Math. Phys. **246**, 503–541 (2004)
31. Lions, P.-L.: Compactness in Boltzmann's equation via Fourier integral operators and applications I. J. Math. Kyoto Univ. **34**, 391–427 (1994)
32. Lions, P.-L.: Compactness in Boltzmann's equation via Fourier integral operators and applications II. J. Math. Kyoto Univ. **34**, 429–461 (1994)
33. Lions, P.-L.: Compactness in Boltzmann's equation via Fourier integral operators and applications III. J. Math. Kyoto Univ. **34**, 539–584 (1994)
34. Lods, B., Toscani, G.: The dissipative linear Boltzmann equation for hard spheres. J. Stat. Phys. **117**, 635–664 (2004)
35. Lu, X.: A direct method for the regularity of the gain term in the Boltzmann equation. J. Math. Anal. Appl. **228**, 409–435 (1998)
36. Martin, P.A., Piaceski, J.: Thermalization of a particle by dissipative collisions. Europhys. Lett. **46**, 613–616 (1999)
37. Mischler, S., Mouhot, C.: Cooling process for inelastic Boltzmann equations for hard spheres, Part II: Self-similar solution and tail behavior. J. Stat. Phys. **124**, 703–746 (2006)
38. Mischler, S., Mouhot, C.: Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres. Commun. Math. Phys. (2007, to appear)
39. Mischler, S., Mouhot, C.: Stability, convergence to the steady state and elastic limit for the Boltzmann equation for diffusively excited inelastic hard spheres. Discrete Continuous Dyn. Syst. Ser. A (2007, to appear)
40. Mischler, S., Wennberg, B.: On the spatially homogeneous Boltzmann equation. Ann. Inst. Henri Poincaré Anal. Non Linéaire **16**, 467–501 (1999)
41. Mischler, S., Mouhot, C., Rodriguez, R.M.: Cooling process for inelastic Boltzmann equations for hard spheres, Part I: The Cauchy problem. J. Stat. Phys. **124**, 655–702 (2006)
42. Mouhot, C.: Etude mathématiques de quelques équations cinétiques collisionnelles. Ph.D. thesis, Ecole Normale Supérieure Lyon, France (2004)
43. Mouhot, C., Villani, C.: Regularity theory for the spatially homogeneous Boltzmann equation with cut-off. Arch. Ration. Mech. Anal. **173**, 169–212 (2004)
44. Spiga, G., Toscani, G.: The dissipative linear Boltzmann equation. Appl. Math. Lett. **17**, 295–301 (2004)
45. Villani, C.: Mathematics of granular materials. J. Stat. Phys. **124**, 781–822 (2006)